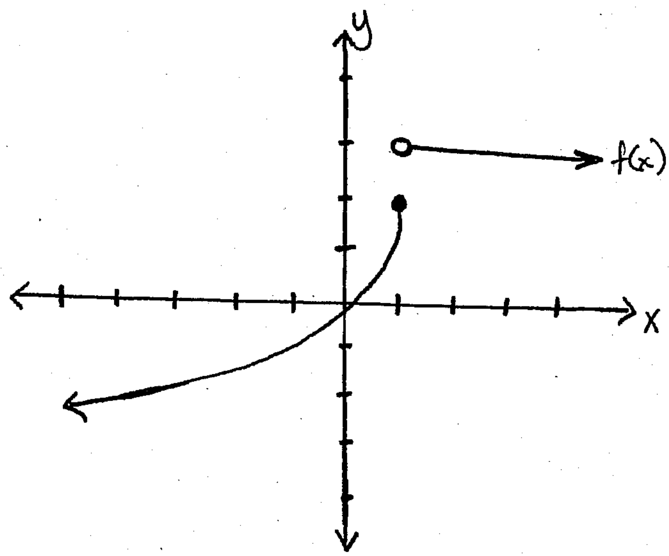
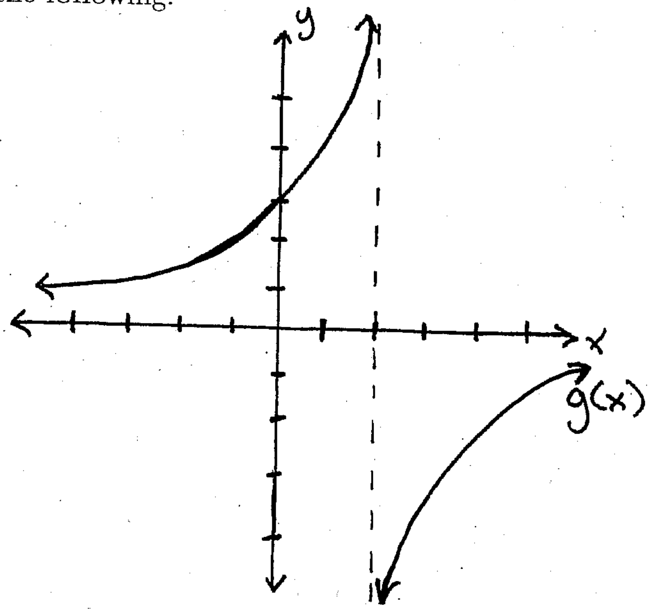
Calculus I 201-NYA-05 C3
Test 1
September 24, 2008

Name: SOLUTIONS

Student Number:

1. *(6 Marks).* Use the graphs to determine the following:





a) $\lim_{x \to 1^-} f(x) = 2$

b) $\lim_{x \to 1^+} f(x) = 3$

c) $f(1) = 2$

d) IS $f(x)$ CONTINUOUS AT $x = 1$? NO

e) $\lim_{x \to 2^-} g(x) = \infty$

f) $\lim_{x \to 2^+} g(x) = -\infty$.

2. Evaluate the following limits:

a) *(3 Marks).* $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 3}$

$$= \frac{(1)^2 + (1) - 2}{(1) - 3} = \frac{0}{-2} = 0$$

b) *(4 Marks).* $\lim_{x \to 3} \frac{x^2 - 9}{x^2 + 2x - 15}$

$$= \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)(x+5)} = \lim_{x \to 3} \frac{(x+3)}{(x+5)}$$

$$= \frac{3+3}{3+5} = \frac{6}{8} = \frac{3}{4}$$

c) *(4 Marks).* $\lim_{x\to-2} \dfrac{x^2+4x+4}{x^2+3x+2}$

$$= \lim_{x\to-2} \frac{(\cancel{x+2})(x+2)}{(\cancel{x+2})(x+1)} = \lim_{x\to-2} \frac{x+2}{x+1} = \frac{-2+2}{-2+1} = \frac{0}{-1} = 0$$

d) *(3 Marks).* $\lim_{x\to\pi/4} 5\sin x$

$$= 5 \sin \frac{\pi}{4}$$

$$= 5 \cdot \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2}$$

e) *(4 Marks).* $\lim_{x\to 2^-} \frac{|x-2|}{x-2}$ $= \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = \lim_{x\to 2^-} -1$

$= -1$

f) *(4 Marks).* $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$

$\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$

$= \lim_{x\to 0} \frac{x+9-9}{x(\sqrt{x+9}+3)} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+9}+3)}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+9}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

g) *(4 Marks).* $\lim_{x\to 5^+} \dfrac{x}{5-x}$

$$\lim_{x\to 5^+} \frac{x}{5-x} = \text{"}\frac{5}{0^-}\text{"} = -\infty$$

h) *(4 Marks).* $\lim_{x\to 2} \dfrac{\sqrt{4x+5}-\sqrt{3x+7}}{x-2}$

$$= \lim_{x\to 2} \frac{\sqrt{4x+5}-\sqrt{3x+7}}{x-2} \cdot \frac{(\sqrt{4x+5}+\sqrt{3x+7})}{(\sqrt{4x+5}+\sqrt{3x+7})}$$

$$= \lim_{x\to 2} \frac{4x+5-(3x+7)}{x-2(\sqrt{4x+5}+\sqrt{3x+7})} = \lim_{x\to 2} \frac{x-2}{x-2(\sqrt{4x+5}+\sqrt{3x+7}}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{4x+5}+\sqrt{3x+7}} = \frac{1}{\sqrt{13}+\sqrt{13}} = \frac{1}{2\sqrt{13}}$$

3. *(7 Marks)*. Use the squeeze theorem to show that:

$$\lim_{x\to 0} |2x| \cos x = 0$$

$$-1 \leq \cos x \leq 1$$

$$\Rightarrow \ -|2x| \leq |2x| \cos x \leq |2x|.$$

$$\lim_{x\to 0} -|2x| = -|2(0)| = 0$$

$$\lim_{x\to 0} |2x| = |2(0)| = 0$$

$$\therefore \lim_{x\to 0} |2x| \cos x = 0$$ BY THE SQUEEZE THM.

4. *(6 Marks)*. Use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to evaluate:

$$\lim_{x\to 0}(5-x)\frac{\sin x}{3x}$$

$$= \lim_{x\to 0} 5\frac{\sin x}{3x} - \lim_{x\to 0} \frac{\cancel{x}\sin x}{3\cancel{x}}$$

$$= \frac{5}{3}\lim_{x\to 0} \frac{\sin x}{x} - \frac{1}{3}\lim_{x\to 0} \sin x$$

$$= \frac{5}{3}(1) - \frac{1}{3}(0) = \frac{5}{3}$$

5. *(6 Marks)*. The function $f(x)$ is defined by

$$f(x) = \begin{cases} 3x^2 - 2x + 1 & \text{if } x < 2 \\ 9x - 1 & \text{if } x \geq 2 \end{cases}$$

Does the limit of $f(x)$ at $x = 2$ exist? If so, what is the value of the limit?

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3x^2 - 2x + 1 = 3(2)^2 - 2(2) + 1$$
$$= 9$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 9x - 1 = 9(2) - 1 = 17$$

$$\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$$

$$\therefore \lim_{x\to 2} f(x) \text{ DNE.}$$

6. *(4 Marks)*. Where is the following function continuous? Clearly explain your answer.

$$g(x) = \frac{x^2 - 25}{x^2 - 7x + 10}$$

CONT. WHERE DENOMINATOR IS NOT 0.

$$x^2 - 7x + 10 = 0 \Rightarrow (x-5)(x-2) = 0$$

$$\therefore x = 5, x = 2$$

$\therefore$ CONTINUOUS EVERYWHERE EXCEPT $x = 5, 2$

7. *(10 Marks).* **a)** State the 3 conditions that a function must satisfy to in order to be continuous at a point $x = a$.

1) $f(a)$ IS DEFINED
2) $\lim_{x \to a} f(x)$ EXISTS
3) $\lim_{x \to a} f(x) = f(a)$

b) Use part **a)** to determine the value of k that makes the following function continuous:

$$f(x) = \begin{cases} \frac{1}{2}x + k & \text{if } x < 4 \\ x^2 - 5k + 4 & \text{if } x \geq 4 \end{cases}$$

1) $f(4) = 20 - 5k$

2) $\lim_{x \to 4^-} f(x) = \frac{1}{2}(4) + k = 2 + k$

$\lim_{x \to 4^+} f(x) = 4^2 - 5k + 4 = -5k$

NEED $20 - 5k = 2 + k$

$\frac{18}{6} = \frac{6k}{6}$ $k = 3.$

3) IF $k = 3$ THEN $f(4) = \lim_{x \to 4} f(x)$

f IS CONT. EVERYWHERE ELSE SINCE IT IS A POLYNOMIAL FOR $x \neq 4$

$\therefore$ f IS CONTINUOUS IF $k = 3.$

8. *(8 Marks)*. Find the vertical asymptotes of the following function. Determine the behavior of the function around the vertical asymptotes.

$$h(x) = \frac{x+2}{x^2 - x - 6}$$

Possible V.A's. $x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3, x = -2$

$x = -2$ gives "$\frac{0}{0}$": $\lim_{x \to -2} \frac{x+2}{(x-3)(x+2)} = \lim_{x \to -2} \frac{1}{x-3}$

$= \frac{1}{-2-3} = \frac{1}{-5}$

$\therefore$ NOT A V.A.

$x = 3$ gives "$\frac{5}{0}$" $\therefore$ V.A.

$\lim_{x \to 3^-} h(x) = \text{"}\frac{7}{0^-(5)}\text{"} = -\infty$

$\lim_{x \to 3^+} f(x) = \text{"}\frac{7}{0^+(5)}\text{"} = \infty$

9. 8 (*Marks*). **a)** Use the definition of the derivative to find the derivative of $f(x) = 3x^2 - 5$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[3(x+\Delta x)^2 - 5] - [3x^2 - 5]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 6x\Delta x + 3(\Delta x)^2 \cancel{-5} \cancel{-3x^2} \cancel{+5}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(6x + 3\Delta x)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} 6x + 3\Delta x = 6x + 3(0) = 6x$$

b) Find of the slope of the tangent line to $f(x)$ at $(1, -2)$ and $(-2, 7)$.

$m_1 = f'(1) = 6(1) = 6$

$m_2 = f'(-2) = 6(-2) = -12$