Calculus I 201-NYA-05 C3
Test 2
October 25, 2008

Name: SOLUTIONS

Student Number:

1. Find the derivative.

a) *(2 marks).*

$$y = x^4 + 3x^3 - 4x^2 + \pi$$

$$y' = 4x^3 + 9x^2 - 8x$$

b) *(2 marks).*

$$y = \frac{2x^{\frac{7}{3}}}{5x^2} = \frac{2}{5}x^{\frac{1}{3}}$$

$$y' = \frac{2}{5} \cdot \frac{1}{3} x^{-\frac{2}{3}} \qquad \frac{2}{15x^{\frac{2}{3}}}$$

c) *(3 marks).*

$$y = (x^3 - 3x^4 + 2)(x^5 + 5x^4 - 3x^2 + 6x - 2)$$

$$y' = (3x^2 - 12x^3)(x^5 + 5x^4 - 3x^2 + 6x - 2) + (x^3 - 3x^4 + 2)(5x^4 + 20x^3 - 6x + 6)$$

d) *(3 marks).*

$$y = \sin(3x + 2)$$

$y' = \cos(3x+2) \cdot (3)$

$= 3\cos(3x+2)$

e) *(4 marks).*

$$y = (x^3 + 1)\sqrt[3]{x^2 + 2} = (x^3+1)(x^2+2)^{\frac{1}{3}}$$

$y' = (3x^2)(x^2+2)^{\frac{1}{3}} + (x^3+1)\frac{1}{3}(x^2+2)^{-\frac{2}{3}}(2x)$

f) *(4 marks).*

$$y = \left(\frac{6x^2 - 5}{4x + 3}\right)^4$$

$$y' = 4\left(\frac{6x^2-5}{4x+3}\right)^3 \frac{d}{dx}\left(\frac{6x^2-5}{4x+3}\right)$$

$$= 4\left(\frac{6x^2-5}{4x+3}\right)^3 \cdot \frac{12x(4x+3) - 4\cdot(6x^2-5)}{(4x+3)^2}$$

g) *(4 marks).*

$$y = \cos^2(3x + 2) = [\cos(3x+2)]^2$$

$$y' = 2\cos(3x+2) \cdot \frac{d}{dx}[\cos(3x+2)]$$

$$= 2\cos(3x+2)(-\sin(3x+2) \cdot (3))$$

2. *(5 marks).* Find the tangent line to the graph of $f(x) = \frac{6x-5}{x^2+1}$ at the point $(1, \frac{1}{2})$.

$$f'(x) = \frac{6(x^2+1) - 2x(6x-5)}{(x^2+1)^2}$$

$$= \frac{6x^2+6-12x^2+10x}{(x^2+1)^2} = \frac{-6x^2+10x+6}{(x^2+1)^2}$$

$$f'(1) = \frac{-6(1)^2+10(1)+6}{(1^2+1)^2} = \frac{10}{4} = \frac{5}{2}$$ SLOPE OF TANGENT LINE AT $x=1$

$\therefore$ $y = mx+b$

$\frac{1}{2} = \frac{5}{2}(1) + b$

$\frac{1}{2} - \frac{5}{2} = b$

$b = -\frac{4}{2} = -2$

$\therefore$ EQUATION OF TANGENT LINE:

$y = \frac{5}{2}x - 2$

3. *(10 marks).* Given $x^2y + y^2 = 16$ use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Express your answer in terms of x and y only.

$$\frac{d}{dx}\left[x^2y + y^2\right] = \frac{d}{dx}[16]$$

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2+2y) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2+2y}$$

$$\frac{d^2y}{dx} = \frac{d}{dx}\left[\frac{-2xy}{x^2+2y}\right]$$

$$= \frac{\frac{d}{dx}[-2xy](x^2+2y) - (-2xy)\frac{d}{dx}[x^2+2y]}{(x^2+2y)^2}$$

$$= \frac{\left(-2y - 2x\frac{dy}{dx}\right)(x^2+2y) + 2xy\left(2x + 2\frac{dy}{dx}\right)}{(x^2+2y)^2}$$

$$= \frac{\left[-2y - 2\left(\frac{-2xy}{x^2+2y}\right)\right](x^2+2y) + 2xy\left[2x + 2\left(\frac{-2xy}{x^2+2y}\right)\right]}{(x^2+2y)^2}$$

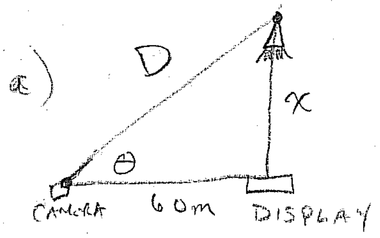
4. *(10 marks)*. A television camera is positioned 60m from where a fireworks display is being set up on the ground. One firework is shot into the air from the display and rises vertically. Its speed is ~~100m~~ 90m/s at the moment it is 80m in the air.

a) ~~How fast is the firework rising at the moment it is 80m in the air?~~

b) If the camera is always kept focused on the firework, how fast is the camera's angle of elevation changing at the moment the firework is 80m in the air?

a) HOW FAST IS THE DISTANCE FROM THE CAMERA TO THE FIREWORK CHANGING AT THE MOMENT IT IS 80m IN THE AIR.

a)



$\frac{dx}{dt} = 90\text{ m/s}$ $\qquad \frac{dD}{dt} = ?$ WHEN $x = 80\text{m}$

$D^2 = x^2 + 60^2$

$D^2 = x^2 + 3600$

DIFFERENTIATING WITH RESPECT TO t:

$2D \cdot \frac{dD}{dt} = 2x \cdot \frac{dx}{dt}$ $\qquad$ NEED D! $\qquad$ WHEN $x = 80$

$D^2 = 80^2 + 60^2 = 10000$

$D = 100\text{ m}$

$\therefore 2(100)\frac{dD}{dt} = 2(80)(90)$

$\boxed{\frac{dD}{dt} = 72\text{ m/s}}$

b) $\frac{d\theta}{dt} = ?$ WHEN $x = 80\text{m}$

$\tan\theta = \frac{x}{60} \Rightarrow 60\tan\theta = x \quad \therefore 60\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$

WHEN $x = 80$, $\cos\theta = \frac{60}{100} = \frac{3}{5} \Rightarrow \sec\theta = \frac{5}{3}$

$\therefore 60\left(\frac{5}{3}\right)^2 \frac{d\theta}{dt} = 90 \Rightarrow \frac{d\theta}{dt} = \frac{90}{60} \cdot \frac{9}{25} = \boxed{\frac{27}{50}\text{ rads/s}}$

MAXIMUM AND

5. *(6 marks)*. Find the absolute ^minimum of the function $f(x) = \sqrt{3x^4 + 4}$ $= (3x^4+4)^{\frac{1}{2}}$ on the interval $[-2, 3]$.

$f'(x) = \frac{1}{2}(3x^4+4)^{-\frac{1}{2}}(12x^3)$

$= \dfrac{6x^3}{(3x^4+2)^{\frac{1}{2}}}$ $\therefore$ CRITICAL NUMBERS ARE

WHERE $f'(x)=0 \Rightarrow x=0$. $f'(x)$ EXIST FOR ALL REAL NUMBERS

SO $x=0$ IS THE ONLY C.N.

$f(0) = \sqrt{3(0)^4+4} = 2$

$f(-2) = \sqrt{3(-2)^4+4} = \sqrt{52} = 2\sqrt{13} \approx 7.21$

$f(3) = \sqrt{3(3)^4+4} = \sqrt{247} \approx 15.72$

$\therefore$ ABSOLUTE MAXIMUM IS $f'(3) = \sqrt{247}$ AND

ABSOLUTE MINIMUM IS $f(0) = 2$

6. *(5 marks)*. Use the quotient rule to show that

$$\frac{d}{dx}[\cot x] = -\csc^2 x.$$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{(-1)(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

Bonus Question. *(4 marks).* Find $f^{(20)}$ (the 20th derivative) where $f(x) = (1+x)^{-1}$.

$$f'(x) = -1(1+x)^{-2}$$

$$f''(x) = (-1)(-2)(1+x)^{-3}$$

$$f^{(3)}(x) = (-1)(-2)(-3)(1+x)^{-4}$$

$$f^{(4)}(x) = (-1)(-2)(-3)(-4)(1+x)^{-5}$$

$$= (-1)^{4}\, 4!\,(1+x)^{-5}$$

$$\therefore\ f^{(n)}(x) = (-1)^{n}\, n!\,(1+x)^{-1-n}$$

$$f^{(20)}(x) = (-1)^{20}\, 20!\,(1+x)^{-1-20}$$

$$= 20!\,(1+x)^{-21}$$