Calculus 201-NYA-05 C3
Test 3
November 22, 2008

Name: SOLUTIONS
Student ID: ______________________

1. Evaluate the following limits (show your work):

a) *(3 marks).*

$$\lim_{x\to\infty} \frac{2x^3+2x-6}{4x^3-3x^2+x}$$

$$= \lim_{x\to\infty} \frac{\frac{2x^3+2x-6}{x^3}}{\frac{4x^3-3x^2+x}{x^3}} = \lim_{x\to\infty} \frac{2+\frac{2}{x^2}-\frac{6}{x^3}}{4-\frac{3}{x}+\frac{1}{x^2}} = \frac{2+0-0}{4-0+0} = \frac{2}{4} = \frac{1}{2}$$

b) *(3 marks).*

$$\lim_{x\to-\infty} \frac{2x^2+6x+1}{7x^4-12}$$

$$= \lim_{x\to-\infty} \frac{\frac{2x^2+6x+1}{x^4}}{\frac{7x^4-12}{x^4}} = \lim_{x\to-\infty} \frac{\frac{2}{x^2}+\frac{6}{x^3}+\frac{1}{x^4}}{7-\frac{12}{x^4}} = \frac{0+0+0}{7-0} = 0$$

c) *(5 marks)*. (Hint: rationalize)

$$\lim_{x\to-\infty}(\sqrt{9x^2+x}+3x)$$

$$= \lim_{x\to-\infty}\left(\sqrt{9x^2+x}+3x\right)\frac{\sqrt{9x^2+x}-3x}{\sqrt{9x^2+x}-3x} = \lim_{x\to-\infty}\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}-3x}$$

$$\lim_{x\to-\infty}\frac{x}{\sqrt{9x^2+x}-3x} = \lim_{x\to-\infty}\frac{\frac{x}{x}}{\frac{\sqrt{9x^2+x}-3x}{x}} = \lim_{x\to-\infty}\frac{1}{\frac{\sqrt{9x^2+x}}{-\sqrt{x^2}}-3}$$

$$= \lim_{x\to-\infty}\frac{1}{-\sqrt{9+\frac{x}{x^2}}-3} = \frac{1}{-\sqrt{9}-3} = \frac{1}{-6}$$

2. *(5 marks)*. Given $f(x) = x\sqrt{4-x^2}$ find all values of x where $f'(x) = 0$ and $f'(x)$ does not exist:

$$f(x) = x(4-x^2)^{\frac{1}{2}}, \quad f'(x) = (4-x^2)^{\frac{1}{2}} + x\,\frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$= (4-x^2)^{\frac{1}{2}} - \frac{x^2}{(4-x^2)^{\frac{1}{2}}}$$

$$= \frac{(4-x^2)}{(4-x^2)^{\frac{1}{2}}} - \frac{x^2}{(4-x^2)^{\frac{1}{2}}} = \frac{4-2x^2}{(4-x^2)^{\frac{1}{2}}}$$

$$= \frac{2(2-x^2)}{(4-x^2)^{\frac{1}{2}}}$$

$\therefore$ $f'(x)=0$ WHEN $x = \pm\sqrt{2}$

$f'(x)$ DOES NOT EXIST WHEN $x > 2$ AND $x < -2$

3. *(7 marks)*. Use the second derivative test to find and classify the relative extrema of the function $f(x) = \cancel{4x^3 - 4x^2 - 24x}$.

$$x^4 - \frac{4}{3}x^3 - 12x^2 + 6$$

$$\therefore \quad f'(x) = 4x^3 - 4x^2 - 24x$$
$$= 4x(x^2 - x - 6)$$
$$= 4x(x-3)(x+2) = 0$$
$$\therefore \quad x = -2, 0, 3$$

$$f''(x) = 12x^2 - 8x - 24$$

$$f''(-2) = 12(-2)^2 - 8(-2) - 24 = 40 > 0$$

$\therefore$ RELATIVE MIN AT $x = -2$

$$f(-2) = (-2)^4 - \frac{4}{3}(-2)^3 - 12(-2)^2 + 6$$
$$= 16 + \frac{32}{3} - 48 + 6 = -26 + \frac{32}{3} = -\frac{78}{3} + \frac{32}{3}$$
$$= -\frac{46}{3} \qquad \therefore \left(-2, -\frac{46}{3}\right) \text{ RELATIVE MIN.}$$

$f''(0) = -24 < 0 \quad \therefore$ RELATIVE MAX AT $x = 0$.

$f(0) = 6 \quad \therefore$ RELATIVE MAX $(0, 6)$.

$f''(3) = 12(3)^2 - 8(3) - 24 = 60 > 0 \quad \therefore$ RELATIVE MIN AT $x = 3$.

$$f(3) = (3)^4 - \frac{4}{3}(3)^3 - 12(3)^2 + 6$$
$$= -57$$

$\therefore$ RELATIVE MIN: $(3, -57)$

4. *(15 marks).* Sketch the graph of f given that:

$$f(x) = \frac{-2x}{x^2-1} \qquad f'(x) = \frac{2(x^2+1)}{(x^2-1)^2} \qquad f''(x) = \frac{-4x(x^2+3)}{(x^2-1)^3}$$

Clearly identify:
a)The domain and x and y intercepts
b)Any symmetry that the graph has
c)Any horizontal and vertical asymptotes
d)The intervals where the function is increasing and decreasing
e)Any relative extrema
f)The intervals where the function is concave upward and concave downward
g)Any points of inflection

Label the graph with all relevant information.

a) DOMAIN: ALL REALS EXCEPT $x = \pm 1$

x & y-int: $(0,0)$

b) $f(-x) = \frac{-2(-x)}{(-x)^2-1} = \frac{2x}{x^2-1} = -f(x)$ ∴ f IS ODD

c) $\lim\limits_{x\to\infty} \frac{-2x}{x^2-1} = \lim\limits_{x\to\infty} \frac{\frac{-2x}{x^2}}{\frac{x^2-1}{x^2}} = \lim\limits_{x\to\infty} \frac{-\frac{2}{x}}{1-\frac{1}{x^2}} = \frac{0}{1-0} = 0$

$\lim\limits_{x\to-\infty} \frac{-2x}{x^2-1} = \lim\limits_{x\to-\infty} \frac{-\frac{2}{x}}{1-\frac{1}{x^2}} = \frac{0}{1-0} = 0$

∴ H.A. $y=0$

* VERTICAL ASYMPTOTES → NEXT PAGE

d) $f'(x) \neq 0$ $f'(x)$ DOES NOT EXIST WHEN $x = \pm 1$

INTERVAL	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
SIGN OF $f'(x)$	+	+	+
CONCLUSION	INCREASING	INCREASING	INCREASING

∴ THE FUNCTION IS INCREASING ON $(-\infty,-1)$, $(-1,1)$ AND $(1,\infty)$

e) NO RELATIVE EXTREMA

POSSIBLE V.A'S WHEN $x = \pm 1$

$\lim_{x\to 1^-} \frac{-2x}{x^2-1} = "\frac{-2}{0^-}" = \infty$ $\quad \lim_{x\to 1^+} \frac{-2x}{x^2-1} = "\frac{-2}{0^+}" = -\infty$ $\quad \therefore$ VA: $x = 1$

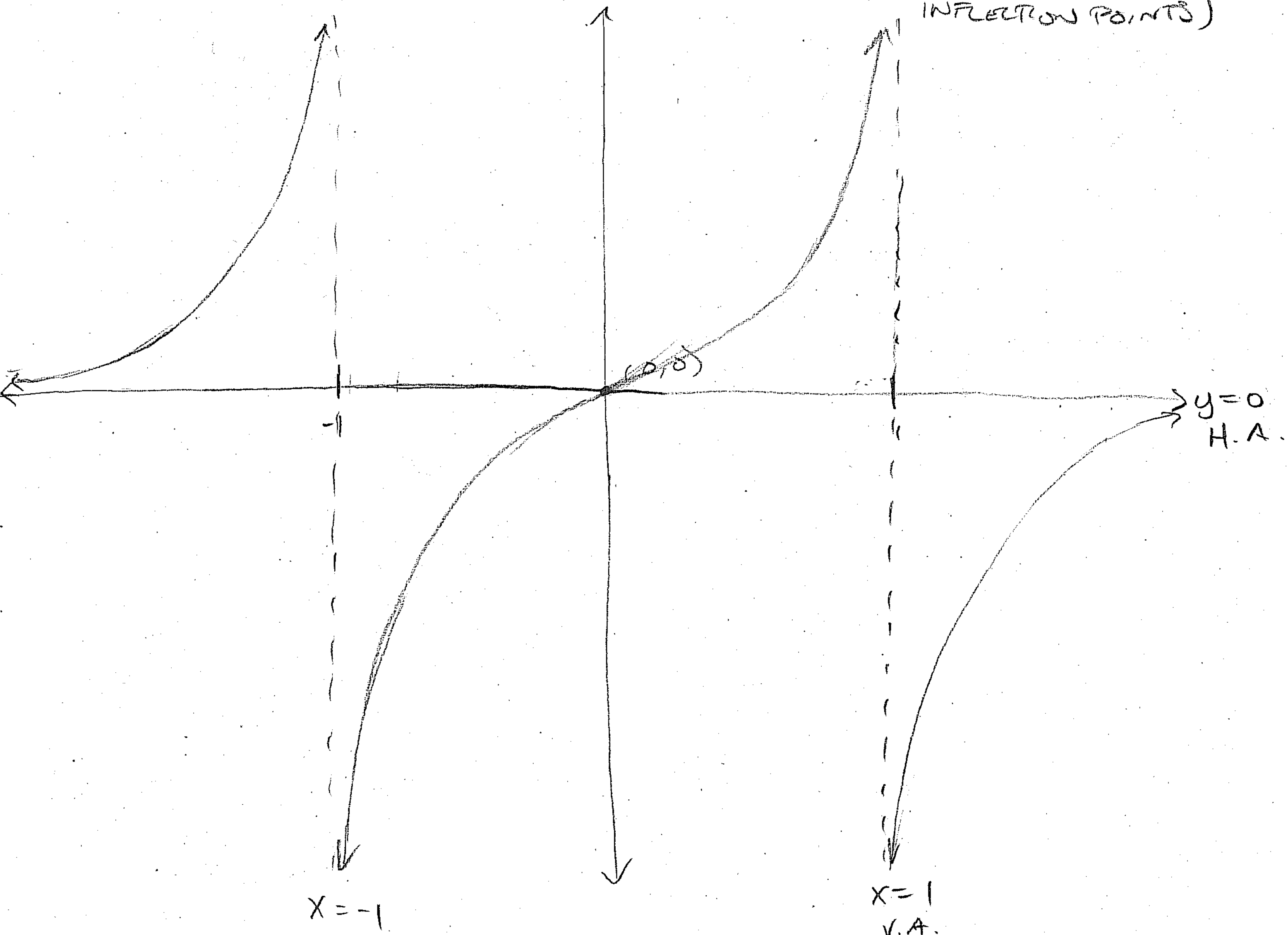
$\lim_{x\to -1^-} \frac{-2x}{x^2-1} = "\frac{2}{0^+}" = \infty$ $\quad \lim_{x\to -1^+} \frac{-2x}{x^2-1} = "\frac{2}{0^-}" = -\infty$ $\quad \therefore$ VA: $x = -1$

) $f''(x) = 0$ WHEN $x = 0$, $f''(x)$ DOES NOT EXIST WHEN $x = \pm 1$

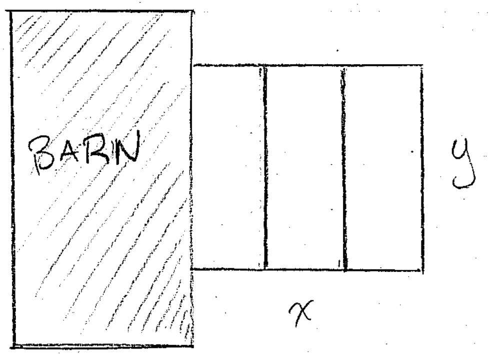
INTERVALS	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
SIGN OF $f''(x)$	+	−	+	−
CONCLUSION	CONCAVE UPWARD	CONCAVE DOWNWARD	CONCAVE UPWARD	CONCAVE DOWNWARD

$\therefore$ f IS CONCAVE UPWARD ON $(-\infty, -1)$ AND $(0, 1)$ AND
CONCAVE DOWNWARD ON $(-1, 0)$ AND $(1, \infty)$

)) INFLECTION POINT: $(0,0)$ (NOTE. $x = \pm 1$ DO NOT YIELD INFLECTION POINTS)



5. *(7 marks).* A farmer intends to fence a rectangular field. To save money on fencing he decides to put the fence next to a barn. The field will be divided into three equal parts as shown in the figure below. If the farmer has 600m of fencing what should the dimensions of the fence be to maximize the area enclosed by the fence? Note: there is no fence used along the side of the barn.



$A = xy$, $\quad P = 3y + 2x = 600m \Rightarrow 3y = 600 - 2x$

$y = 200 - \frac{2}{3}x$

$\therefore A(x) = x(200 - \frac{2}{3}x) = 200x - \frac{2}{3}x^2$

$A'(x) = 200 - \frac{4}{3}x$

$A'(x) = 0 \Rightarrow \quad 0 = 200 - \frac{4}{3}x$

$\frac{4}{3}x = 200$

$x = 150$

$A''(x) = -\frac{4}{3}$

$\therefore A''(150) = -\frac{4}{3} < 0$

$\therefore x = 150$ GIVES MAXIMUM AREA

$y = 200 - \frac{2}{3}(150) = 100$

$\therefore \quad x = 150m \qquad y = 100m$

6. Find the following indefinite integrals:

a) *(2 marks).*

$$\int (x^2 + 12x - 5)\ dx$$

$$= \frac{x^3}{3} + \frac{12x^2}{2} - 5x + c$$

$$= \frac{1}{3}x^3 + 6x^2 - 5x + c$$

b) *(3 marks).*

$$\int \frac{x^4 + 7x^3 - 4}{x^2}\ dx$$

$$= \int \left(x^2 + 7x - \frac{4}{x^2}\right) dx = \int \left(x^2 + 7x - 4x^{-2}\right) dx$$

$$= \frac{x^3}{3} + \frac{7x^2}{2} - \frac{4x^{-1}}{-1} + c = \frac{1}{3}x^3 + \frac{7}{2}x^2 + \frac{4}{x} + c$$

c) *(3 marks).*

$$\int \sqrt{x}(3 + x)\ dx$$

$$= \int x^{\frac{1}{2}}(3 + x)dx = \int \left(3x^{\frac{1}{2}} + x^{\frac{3}{2}}\right)dx = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= \frac{2}{3} \cdot 3x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$$

$$= 2x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C$$

d) *(2 marks).*

$$\int (5\sin x + \sec^2 x)\, dx$$

$$= -5\cos x + \tan x + C$$

7. *(4 marks).* Use a change in variables to find the following integral:

$$\int \frac{x^2}{\sqrt{1+4x^3}}\, dx$$

LET $u = 1 + 4x^3$

$du = 12x^2\, dx$

$$\int \frac{x^2}{\sqrt{1+4x^2}}\, dx$$

$$= \int \frac{\cancel{x^2}}{\sqrt{u}} \frac{du}{12\cancel{x^2}}$$

$$= \frac{1}{12} \int \frac{1}{u^{\frac{1}{2}}}\, du$$

$$= \frac{1}{12} \int u^{-\frac{1}{2}}\, du$$

$$= \frac{1}{12} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{1} \cdot \frac{1}{12} u^{\frac{1}{2}} + C$$

$$= \frac{1}{6} u^{\frac{1}{2}} + C = \frac{1}{6}(1+4x^3)^{\frac{1}{2}} + C$$