FRACTIONS AND DECIMALS

A common fraction represents a part of the whole. Thus ¾ means 3 parts out of a whole of 4. The number above the dividing line is called the numerator. The number below is called the denominator. Thus in the fraction ¾ the numerator is 3 and the denominator is 4. An improper fraction is a fraction in which the numerator is greater than or equal to the denominator; e.g. $\frac{5}{3}$, $\frac{2}{2}$.

If we divide (or multiply) the numerator and denominator of a fraction by the same number we obtain an equivalent fraction. We use the equal sign to indicate equivalence of fractions. Thus $\frac{2}{4}$ is equivalent to $\frac{1}{2}$ since $\frac{2\div 2}{4\div 2}=\frac{1}{2}$ and we write $\frac{2}{4}=\frac{1}{2}$.

Similarly $\frac{9}{12}=\frac{9\div 3}{12\div 3}=\frac{3}{4}$ and we write $\frac{9}{12}=\frac{3}{4}$. We say a fraction is in lowest terms if there is no positive integer greater than one (2, 3, 4, ...) which divides both the numerator and denominator.

Thus $\frac{6}{10}$ is not in lowest terms. Since $\frac{6\div 2}{10\div 2}=\frac{3}{5}$ whereas $\frac{3}{5}$ is in lowest terms. If a fraction is equivalent to another fraction in lowest terms we say the fraction has been reduced to lowest terms.

Example 1) Reduce $\frac{25}{10}$ to lowest terms $\frac{25\div 5}{10\div 5}=\frac{5}{2}$.

Thus $\frac{25}{10}=\frac{5}{2}$ and $\frac{5}{2}$ is in lowest terms.

FRACTIONS can be converted to decimals by performing the indicated division. We stop if either the division is exact or a pattern emerges.

Thus $\frac{3}{4}=.75$ since $4\overline{)3.0}$ with quotient $.75$

whereas $\frac{1}{3}=.333\cdots$ since $3\overline{)1.0}$ with quotient $.33\ldots$

which can be written as $.\dot{3}$

ie. $\frac{1}{3} = .\dot{3}$ where the dot indicates that the 3 repeats.

Example 2) Express $\frac{2}{7}$ as a repeating decimal we divide 7 into 2

$$7\overline{)\ 2.00000}\quad .2857428...$$

Thus $\frac{2}{7} = .\overline{28574}$

The bar indicates that 28574 repeats an infinite number of times.

Thus we see that $\frac{3}{4}$ is a terminating decimal and $\frac{1}{3}$ and $\frac{2}{7}$ are repeating decimals. A mixed number is a fraction consisting of a whole number and a proper fraction. Thus $2\frac{1}{4}$ is a mixed number. We can convert the mixed number to an improper fraction by multiplying the whole number by the denominator adding the numerator, and putting the result over the denominator. Thus

$2\frac{1}{4} = \frac{2(4)+1}{4} = \frac{9}{4}$ $\quad$ $8\frac{2}{3} = \frac{(8\times 3)+2}{3} = \frac{26}{3}$ (improper fraction) and

$5\frac{2}{5} = \frac{5(5)+2}{5} = \frac{27}{5}$.

We can convert an improper fraction to a mixed number by doing the division and writing the result as the quotient plus the remainder divided by the divisor.

Thus $\frac{24}{5} = 4\frac{4}{5}$ since $5\overline{)\ 24}$ 4R4

$\frac{30}{9} = 3\frac{3}{9}$ or $3\frac{1}{3}$ since $9\overline{)\ 30}$ 3R3

A mixed number can be converted into a decimal by writing it as the sum of the whole number and the decimal equivalent of the proper fraction.

Thus $5\frac{1}{2} = 5 + \frac{1}{2} = 5.5$

$2\frac{3}{4} = 2 + \frac{3}{4} = 2.75$

$6\frac{2}{3} = 6 + \frac{2}{3} = 6.66$ or $6.\dot{6}$

We often have to round decimals (usually to 2 decimal places) by dropping decimal digits. Thus .2379 rounded to 2 decimal places requires that we drop the 79.

Procedure for Rounding

1) If the first digit of the group of decimal digits dropped is 5, 6, 7, 8, or 9 we add 1 to the last retained digit. Hence .2379 is rounded to .24

2) If the first digit of the group of the decimal digits dropped is 0, 1, 2, 3 or 4 we do not change the last retained decimal digit. Thus .24329 is rounded to .24
(2 decimal places).

Multiplying and Dividing Fractions

We multiply 2 fractions by multiplying their respective numerators and denominators and reducing to lowest terms. Thus $\frac{2}{3} \times \frac{4}{9} = \frac{8}{27}$ which is in lowest terms. We divide 2 fractions by multiplying the first fraction by the reciprocal of the second.

Thus $\frac{2}{5} \div \frac{4}{7} = \frac{2}{5} \times \frac{7}{4} = \frac{14}{20} = \frac{7}{10} = .7$ or if we write $\frac{\frac{3}{4}}{\frac{2}{3}}$ we first write

$$\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8} = 1.125$$

ADDITION and SUBTRACTION of FRACTIONS

1. Like Denominators.

If the denominators are the same we simply add or subtract the respective numerators.

Ex. 1) $\frac{2}{5}+\frac{4}{5}=\frac{6}{5}=1\frac{1}{5}$.

Ex. 2) $\frac{3}{7}-\frac{5}{7}=-\frac{2}{7}$

If mixed numbers are present, convert the mixed number to an improper fraction and then perform the required operation.

Ex. 3) $1\frac{2}{3}-4\frac{1}{3}=\frac{5}{3}-\frac{13}{3}=-\frac{8}{3}=-2\frac{2}{3}$

2. Unlike Denominators.

If the denominators are different, find the common denominator and adjust the individual denominators so that the original fractions are replaced by equivalent fractions with the common denominator. Then perform the indicated operations.

Ex. 1) $\frac{1}{3}+\frac{1}{2}=?$

Clearly the 2 fractions have different denominators a common denominator of 2 and 3 is 6.

Thus $\frac{1}{3}=\frac{1\times 2}{3\times 2}=\frac{2}{6}$ (equivalent fraction).

Since the denominator was multiplied by 2 the numerator must also be multiplied by 2.

Similarly $\frac{1}{2}=\frac{1\times 3}{2\times 3}=\frac{3}{6}$.

Notice again that since we multiplied the 2 by 3 in the denominator we must multiply the 1 by 3 in the numerator.

Now $\frac{1}{3}+\frac{1}{2}=\frac{2}{6}+\frac{3}{6}=\frac{5}{6}$

Finding a common denominator.

Method 1) One method of finding a common denominator is to write out successive multiples of each denominator. The smallest or least common multiple (L.C.M.) is the least common denominator (L.C.D.).

Thus for $\frac{1}{2}$ and $\frac{1}{3}$

The multiples of 2 are $\{2, 4, 6, \ldots\}$

And the multiples of 3 are $\{3, 6, 9, \ldots\}$

The least common multiple and hence L.C.D. is 6.

Sometimes this method involves a fair amount of work.

Thus for $\frac{1}{8}+\frac{1}{18}$ we need the L.C.M. of 8 and 18.

The multiples of 8 are $\{8, 16, 24, 32, 40, 48, 56, 64, \underline{72}, 80, 88 \ldots\}$

The multiples of 18 are $\{18, 36, 54, \underline{72}, 90, 108 \ldots\}$

The L.C.M. and hence L.C.D. is 72.

Hence $\frac{1}{8}+\frac{1}{18}=\frac{1\times 9}{8\times 9}+\frac{1\times 4}{18\times 4}=\frac{9+4}{72}=\frac{13}{72}$.

Method 2) In order to discuss method 2 we need to define a prime number. By "number" we mean numbers of the form $\{1, 2, 3, \ldots\}$ sometimes called natural numbers or positive integers.
Now a number is said to be prime if its only (positive) factors are one and itself. Otherwise a number is said to be composite.
Thus 2 and 3 are prime since their only factors are respectively $\{1, 2\}$ and $\{1, 3\}$ but 6 is composite since its factors are $\{1, 2, 3, 6\}$. In fact all even numbers greater than 2 are composite. (Why?) An important result in mathematics states that every positive number greater than 1 can be written as a product of powers of prime factors.

Ex. 1) $14 = 2^1 \times 7^1$

Here the prime factors are 2 and 7 and the respective powers are both 1.

Ex. 2) $50 = 2\times 25 = 2^1 \times 5^2$

Here the prime factors are 2 and 5. The respective powers are 1 and 2.

Ex. 3) $72 = 8\times 9 = 2^3 \times 3^2$.

The prime factors are 2 and 3. The respective powers are 3 and 2.

Ex. 4) $256 = 16^2 = \left(2^4\right)^2 = 2^8$.

We have only prime factor 2. Its power is 8.

Given two fractions we can find the L.C.D. of their denominators by:

1) expressing each denominator as a product of powers of prime numbers.

2) taking each prime factor (in both numbers) and raising it to its highest power.

3) multiplying all the prime factors raised to their powers obtained in step 2. The resulting number is the required L.C.D.

Ex. 1) Consider again $\frac{1}{8}+\frac{1}{18}$

i) $8=2^3, \quad 18=2\times 9=2\times 3^2$

ii) The highest power of 2 is 3, so take 2^3.
The highest power of 3 is 2, so take 3^2

iii) The required L.C.D. is $2^3\times 3^2=8\times 9=72$

As before $\frac{1}{8}+\frac{1}{18}=\frac{9}{72}+\frac{4}{72}=\frac{13}{72}$

Ex. 2) Simplify $\frac{11}{50}-\frac{7}{20}$

Method i) Multiples of 50 are {50, 100}
Multiples of 20 are {20, 40, 60, 80, 100}
The L.C.D. is 100.

$$\frac{11}{50}-\frac{7}{20}=\frac{11\times 2}{50\times 2}-\frac{7\times 5}{20\times 5}=\frac{22}{100}-\frac{35}{100}=-\frac{13}{100}$$

Method ii) $50=2\times 25=2\times 5^2$
$20=4\times 5=2^2\times 5$
L.C.D. $=2^2\times 5^2=4\times 25=100$

As before $\frac{11}{50}-\frac{7}{20}=\frac{11\times 2}{50\times 2}-\frac{7\times 5}{20\times 5}=\frac{22-35}{100}=-\frac{13}{100}$

Complex Fractions

A complex fraction is a fraction in which either the numerator, the denominator, or both, are themselves fractions.

Ex. 1) $$\frac{2+\frac{3}{4}}{5}$$

2) $$\frac{3}{5+\frac{1}{3}}$$

3) $$\frac{3-\frac{2}{5}}{4+\frac{1}{3}}$$

We simplify complex fractions either by reducing the respective numerators and denominators to mixed numbers and then dividing or by use of a calculator.

Ex. 1) $$\frac{2+\frac{3}{4}}{5}=\frac{\frac{2(4)+3}{4}}{5}=\frac{\frac{11}{4}}{5}=\frac{11}{4}\div 5=\frac{11}{4}\times\frac{1}{5}=\frac{11}{20}=.55$$

or using the calculator

$$\frac{2+\frac{3}{4}}{5}=\frac{2.75}{5}=.55$$

2) $$\frac{3}{5+\frac{1}{3}}=\frac{3}{\frac{16}{3}}=3\div\frac{16}{3}=3\times\frac{3}{16}=\frac{9}{16}=.5625$$

or $$\frac{3}{5.33...}=.5625$$

3) $$\frac{3-\frac{2}{5}}{4+\frac{1}{3}}=\frac{\frac{15-2}{5}}{\frac{12+1}{3}}=\frac{\frac{13}{5}}{\frac{13}{3}}=\frac{13}{5}\div\frac{13}{3}=\frac{13}{5}\times\frac{3}{13}=\frac{3}{5}=.6$$

or $$\frac{3-\frac{2}{5}}{4+\frac{1}{3}}=\frac{2.6}{4.333...}=.6$$

Percent

A percent is a fraction in which the denominator is 100. Percent means per hundred. We use the symbol % to indicate parts of 100.

Thus 1) $\frac{20}{100}$ fraction

= 2) .20 decimal

= 3) 20%

In general a decimal can be converted to a percent by shifting the decimal 2 units to the right and a percent can be converted to a decimal by shifting the decimal 2 units to the left.

Thus $.27 = 27\%$ but $52\% = .52$

A fraction is converted to a percent by dividing the numerator by the denominator and multiplying by 100.

Thus $\frac{3}{4} = .75 = 75\%$

$\frac{8}{5} = 1.6 = 160\%$.

Conversely a percent can be converted to a fraction by dividing the percent by 100 and reducing the resulting fraction to lowest terms.

Ex. 1) $65\% = \frac{65}{100} = \frac{13}{20}$

2) $125\% = \frac{125}{100} = \frac{5}{4}$

3) $.8\% = \frac{.8}{100} = \frac{8}{1000} = \frac{1}{125}$

4) $\frac{1}{4}\% = .25\% = \frac{.25}{100} = \frac{25}{10000} = \frac{1}{400}$