

RATIONAL EXPRESSIONS (FRACTIONS)

THE DEFINITION OF A RATIONAL EXPRESSION

A FRACTION $\frac{P}{Q}$, where $Q \neq 0$, such that the numerator, P , and the denominator, Q , are polynomials.

THE FUNDAMENTAL PRINCIPLE OF FRACTIONS

$$\frac{PK}{QK} = \frac{P}{Q} \quad \text{where } Q, K \neq 0$$

OPERATIONS

MULTIPLICATION

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

DIVISION

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P \cdot S}{Q \cdot R} = \frac{PS}{QR}$$

ADDITION AND SUBTRACTION

LIKE DENOMINATORS

$$\frac{P}{Q} \pm \frac{R}{Q} = \frac{P \pm R}{Q}$$

UNLIKE DENOMINATORS

Use the LEAST COMMON DENOMINATOR (LCD) of the fractions to rewrite them with like denominators and operate as above.

COMPLEX FRACTIONS

Complex Fractions are fractions whose numerators and/or denominators contain fractions.

To simplify a complex fraction write the numerator and denominator as single fractions and divide as follows:

$$\frac{\frac{P}{Q}}{\frac{R}{S}} = \frac{P}{Q} \div \frac{R}{S} = \frac{P \cdot S}{Q \cdot R}$$

RATIONAL EXPRESSIONS (FRACTIONS) - EXAMPLES

(1) Consider the fraction $\frac{x-1}{x+1}$.

(a) Evaluate it for $x=1$.

consider $\frac{1-1}{1+1} = \frac{0}{2} = 0 \cdot \frac{1}{2} = 0$

(b) Find the value(s) of x for which it is undefined.

consider $x+1=0$
 $\therefore x=-1$, hence undefined for $x=-1$.

(2) Simplify $\frac{x^2-25}{x^2-4x-5} = \frac{(x+5)(\cancel{x-5})}{(x+1)(\cancel{x-5})}$ } The Fundamental
 = $\frac{x+5}{x+1}$ } Principle of
 Fractions

(3) Simplify $\frac{x-y}{y-x} = \frac{(-1)(\cancel{y-x})}{(\cancel{y-x})}$
 = -1

Note: all such ratios of opposites equal -1 .

(4) Multiply and simplify $\frac{x}{(x+1)} \cdot \frac{(x+1)^2}{x^2} = \frac{\cancel{x} \cdot (x+1)(x+1)}{(x+1) \cdot \cancel{x} \cdot x}$
 = $\frac{x+1}{x}$

Note: We may multiply by cancelling common factors across the dot product.

(5) Divide and simplify $\frac{(2-x)}{15} \div \frac{(x-2)}{3} = \frac{(2-x)}{15} \cdot \frac{3}{(x-2)}$
 = $\frac{(2-x)}{5(x-2)}$
 = $\frac{(-1)(\cancel{x-2})}{5(\cancel{x-2})}$
 = $-\frac{1}{5}$

RATIONAL EXPRESSIONS (FRACTIONS) - EXAMPLES

(6) Add and simplify

$$\frac{x}{4x+4} + \frac{1}{4x+4} = \frac{x+1}{4x+4}$$

$$= \frac{1(x+1)}{4(x+1)}$$

$$= \frac{1}{4}$$

(7) Subtract and simplify

$$\frac{x}{2x-4} - \frac{2}{x^2-2x}$$

$$= \frac{x}{2(x-2)} - \frac{2}{x(x-2)} \quad , \text{ LCD is } 2x(x-2)$$

$$= \frac{x^2}{2x(x-2)} - \frac{4}{2x(x-2)}$$

$$= \frac{x^2-4}{2x(x-2)}$$

$$= \frac{(x+2)(x-2)}{2x(x-2)}$$

$$= \frac{x+2}{2x}$$

(8) Simplify the complex fraction

$$\frac{\frac{x^2-9}{3}}{\frac{x}{6} + \frac{1}{2}} = \frac{\frac{x^2-9}{3}}{\frac{x+3}{6}} = \frac{x^2-9}{3} \div \frac{x+3}{6}$$

$$= \frac{(x+3)(x-3)}{3} \cdot \frac{6}{(x+3)} = 2(x-3)$$

Alternate Method: Multiply numerator and denominator by 6
(the LCD of all the fractions)

$$\frac{6\left(\frac{x^2-9}{3}\right)}{6\left(\frac{x}{6} + \frac{1}{2}\right)} = \frac{2(x^2-9)}{6 \cdot \frac{x}{6} + 6 \cdot \frac{1}{2}} = \frac{2(x+3)(x-3)}{(x+3)} = 2(x-3)$$

RATIONAL EXPRESSIONS (FRACTIONS) - EXERCISES

① Evaluate each fraction for $x = -2$, and find the value(s) of x for which each fraction is undefined.

(a) $\frac{5}{3x}$

(b) $\frac{x+2}{x+3}$

(c) $\frac{3x^2-1}{2x-8}$

(d) $\frac{x^2+2}{x^2-1}$

(e) $\frac{7x^3}{x^2-4x-5}$

(f) $\frac{5}{x^2+1}$

② Simplify:

(a) $\frac{4x^3y^4}{6x^4y}$

(b) $\frac{3xy}{xy+x}$

(c) $\frac{5-2x}{2x-5}$

(d) $\frac{5x-5}{x^3-x^2}$

(e) $\frac{15+5x}{3x+9}$

(f) $\frac{24x^2-54}{6x-9}$

(g) $\frac{x(x-6)+9}{x^2-9}$

(h) $\frac{2x^2+3x-2}{2x-1}$

(i) $\frac{7x^2-31x-20}{7x+4}$

(j) $\frac{x^2+2x-15}{x^2-7x+12}$

(k) $\frac{3x^2+x-2}{3x^2-5x+2}$

(l) $\frac{3x^3-12x}{6x^3-24x^2+24x}$

(m) $\frac{x^3+8}{x+2}$

(n) $\frac{2a^3-16}{2a^2+4a+8}$

(o) $\frac{x^2y+xy+5x^2+5}{5x+xy}$

③ Multiply and simplify:

(a) $\frac{27x^3}{9x} \cdot \frac{24}{9x^2}$

(b) $\frac{64^2}{18x^2} \cdot \frac{9xy}{3y^3}$

(c) $\frac{5-x}{5+x} \cdot \frac{x+5}{x-5}$

(d) $\frac{5x-15}{3x+9} \cdot \frac{4x+12}{6x-18}$

RATIONAL EXPRESSIONS (FRACTIONS) - EXERCISES

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- (e) $\frac{x}{x-1} \cdot \frac{x^2-1}{x^2}$ (f) $\frac{x^2-x}{2x+4} \cdot \frac{x+2}{x}$
- (g) $\frac{x^3-x^2y}{xy} \cdot \frac{3y}{3x-3y}$ (h) $\frac{x^2-4a^2}{ax+2a^2} \cdot \frac{2a}{x-2a}$
- (i) $\frac{x^2+x-20}{x^2+2x-15} \cdot \frac{x^2+4x-21}{x^2+3x-28}$ (j) $\frac{x^2-y^2}{x^2-2xy+y^2} \cdot \frac{y-x}{x+y}$
- (k) $\frac{x^2-1}{3x^2+4x+1} \cdot \frac{9x^2-1}{3x^2-4x+1}$ (l) $\frac{x^2+3x+2}{x^2+5x+4} \cdot \frac{x^2+10x+24}{x^2+5x+6}$
- (m) $\frac{x^2-1}{2x-4} \cdot \frac{x^2-4}{x^2-x-2} \cdot \frac{3x-6}{x^2+x-2}$ (n) $\frac{x+y}{2x^2y+2xy^2} \cdot \frac{x^3+xy^2}{x^4-y^4} \cdot \frac{x^2y-y^3}{y}$

4 Divide and simplify:

- (a) $\frac{(3x)^2}{(2y)^2} \div \frac{6x^3}{16y^2}$ (b) $\frac{2x}{x+4} \div \frac{x^2}{(x+4)^2}$
- (c) $\frac{10x+10}{3x-6} \div \frac{2x+2}{3xy-6y}$ (d) $\frac{15x+15}{x^2-9} \div \frac{15-5x}{(x-3)^2}$
- (e) $\frac{x^2+4x-5}{x^2+7x+10} \div \frac{x-1}{x+4}$ (f) $\frac{x^2-3x-10}{x^2-5x} \div \frac{x^2-4}{x^2-2x}$
- (g) $\frac{x^2+9x+20}{x^2+8x+16} \div \frac{x^2-25}{5x+20}$ (h) $\frac{x^2-4}{x^2-2x} \div \frac{x^2+3x+2}{x^3+2x^2+x}$
- (i) $\frac{16x^2-1}{4x^2+3x-1} \div \frac{4x^2-7x-2}{x^2-x-2}$ (j) $\frac{6x^3+7x^2}{12x-3} \div \frac{6x^2+7x}{36x-9}$
- (k) $\frac{2x^2+8x-42}{x-3} \div \frac{2x^2+14x}{x^2+5x}$ (l) $\frac{x^3-x}{x^2-3x-4} \div \frac{x-x^2}{x^2-16}$
- (m) $\frac{3x^2-2xy-y^2}{3x^2-5xy-2y^2} \div \frac{8x-8y}{4x-8y}$ (n) $\frac{12x^4+15x^2}{15x^2-x-2} \div \frac{4x^2+5}{9x^2-1}$

5 Multiply or divide and simplify:

(a) $\left(\frac{x^2+5x}{x^2-25} \cdot \frac{x^2-x-20}{3x+12} \right) \div \frac{x^2+3x}{3x^2-27}$

RATIONAL EXPRESSIONS (FRACTIONS) - EXERCISES

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$$(b) \left(\frac{6x^2 - x - 2}{x-1} \cdot \frac{x-1}{9x^2 - 4} \right) \div \frac{2x+1}{3x+2}$$

$$(c) \left(\frac{x^2 - x - 20}{x^2 - 25} \cdot \frac{x^2 - x - 2}{x^2 + 2x - 8} \right) \div \frac{x+1}{x^2 + 5x}$$

$$(d) \left(\frac{x^2 - x - 6}{x-2} \div \frac{x^2 - 4x}{x^2 - x - 2} \right) \cdot \frac{x-4}{x^2 + x}$$

$$(e) \frac{y^2}{x+1} \cdot \frac{x^2 + 2x + 1}{x^2 - 1} \div \frac{3y}{xy - y}$$

$$(f) \frac{2x^2 - x}{4x^2 - 1} \cdot \frac{4x^2 + 4x + 1}{3x} \div \frac{4x^2 - 2x - 2}{6x^2 - 6x}$$

$$(g) \frac{4x^2 - 25}{3x + 3} \cdot \frac{7x - 1}{2x^2 - 9x + 10} \div \frac{2x + 5}{3x^2 - 3x - 6}$$

$$(h) \frac{a^2 - ax}{3ax - 2x^2} \cdot \frac{4ax + 2x^2}{ax - x^2} \div \frac{4a^2 + 2ax}{9a - 6x}$$

$$(i) \frac{x^4 - 8x}{x^2 - 4x - 5} \cdot \frac{x^2 + 2x + 1}{x^3 - x^2 - 2x} \div \frac{x^2 + 2x + 4}{x - 5}$$

6 Add or subtract and simplify:

$$(a) \frac{5x}{18} + \frac{7x}{18}$$

$$(b) \frac{4x}{x-6} - \frac{24}{x-6}$$

$$(c) \frac{2-4x}{3-2x} - \frac{4}{2x-3}$$

$$(d) \frac{3x}{x-5} - \frac{2x-25}{5-x}$$

$$(e) \frac{3x+1}{x-7} - \frac{5x+2}{x-7} + \frac{2x+1}{x-7}$$

$$(f) \frac{2x^2}{x-5} - \frac{25+x^2}{x-5}$$

$$(g) \frac{3x^2+2x}{x-1} - \frac{10x-5}{x-1}$$

$$(h) \frac{3x^2}{x^2-1} - \frac{x+4}{x^2-1}$$

$$(i) \frac{3x^2-6}{x^2+x-20} + \frac{x-9}{x^2+x-20} - \frac{2x^2+x+1}{x^2+x-20}$$

RATIONAL EXPRESSIONS (FRACTIONS) - EXERCISES

(7) Add or subtract and simplify:

(a) $\frac{1}{3} - \frac{3-2x}{2}$

(b) $\frac{2x+9}{9x} - \frac{x-5}{5x}$

(c) $\frac{2x-3}{3} + \frac{3x-2}{2} - \frac{4x-1}{5}$

(d) $\frac{x}{x-1} - \frac{2}{x^2-1}$

(e) $\frac{x}{x-2} + \frac{4+2x}{x^2-4}$

(f) $\frac{x}{x^2-4} - \frac{2}{4-x^2}$

(g) $\frac{x}{x-y} - 2 - \frac{x}{y-x}$

(h) $\frac{3x-1}{x^2-10x+25} - \frac{3}{x-5}$

(i) $\frac{2}{x-2} - \frac{2}{x+2} + \frac{3}{x^2-4}$

(j) $\frac{2}{1+x} - \frac{3}{1-x} - \frac{4}{x^2-1}$

(k) $\frac{4x+1}{x-8} - \frac{3x+2}{x+4} - \frac{49x+4}{x^2-4x-32}$

(l) $\frac{x^2-11}{x^2+7x+6} - \frac{x}{x+6} + \frac{2}{x+1}$

(m) $\frac{2-5x}{x+3} - \frac{3+x}{3-x} + \frac{2x(2x-1)}{x^2-9}$

(n) $\frac{1}{1-2x} - \frac{3}{1+2x} + \frac{2(4x-1)}{4x^2-1}$

(8) Simplify the complex fractions:

(a) $\frac{\frac{x+4}{x+1}}{\frac{x+4}{x^2-1}}$

(b) $\frac{\frac{12}{5x+5y}}{\frac{18}{x+y}}$

(c) $\frac{1 + \frac{x}{y}}{\frac{x}{y} - 1}$

(d) $\frac{x - \frac{y^2}{x}}{1 + \frac{y}{x}}$

(e) $\frac{\frac{1}{y} + \frac{1}{x}}{\frac{1}{y^2} - \frac{1}{x^2}}$

(f) $\frac{\frac{1}{7} - \frac{1}{x}}{x-7}$

(g) $\frac{\frac{1}{x} - \frac{1}{x+1}}{1}$

(h) $\frac{\frac{2}{1-x^2}}{\frac{1}{1-x} - \frac{1}{1+x}}$

RATIONAL EXPRESSIONS (FRACTIONS) - EXERCISES

(8)
$$\frac{x - \frac{9}{x}}{x + 4 + \frac{3}{x}}$$

(1)
$$\frac{\frac{3}{a} + \frac{a}{3} - 2}{\frac{a}{6} + \frac{1}{2} - \frac{3}{a}}$$

(f)
$$\frac{x + \frac{a}{x}}{x - \frac{a^4}{x^3}}$$

(2)
$$\frac{1}{1 - \frac{1}{1+a}} + \frac{1}{1 - \frac{1}{1-a}}$$

(M)
$$\frac{2x^{-1} + 4x^{-2}}{2x^{-2} + x^{-1}}$$

(N)
$$\frac{1 - 25x^{-2}}{1 + 10x^{-1} + 25x^{-2}}$$

ANSWERS

(1) (a) $-\frac{5}{6}$ and 0 (b) 0 and -3 (c) $-\frac{1}{2}$ and 4 (d) 2 and ± 1 (e) -8 and -1,5 (f) 1 and none

(2) (a) $\frac{2y^3}{3x}$ (b) $\frac{3y}{y+1}$ (c) -1 (d) $\frac{5}{x^2}$ (e) $\frac{5}{3}$ (f) $2(2x+3)$ (g) $\frac{x-3}{x+3}$ (h) $x+2$ (i) $x-5$ (j) $\frac{x+5}{x-4}$ (k) $\frac{x+1}{x-1}$
 (l) $\frac{x+2}{2(x-2)}$ (m) $x^2 - 2x + 4$ (n) $a-2$ (o) $\frac{x^2+1}{x}$

(3) (a) 8 (b) $\frac{1}{x}$ (c) -1 (d) $\frac{10}{9}$ (e) $\frac{x+1}{x}$ (f) $\frac{x-1}{2}$ (g) x (h) 2 (i) 1 (j) -1 (k) 1 (l) $\frac{x+6}{x+3}$ (m) $\frac{3}{2}$ (n) $\frac{1}{2y}$

(4) (a) $\frac{6}{x}$ (b) $\frac{2(x+4)}{x}$ (c) $5y$ (d) -1 (e) $\frac{x+4}{x+2}$ (f) 1 (g) $\frac{5}{x-5}$ (h) $x+1$ (i) 1 (j) $3x$ (k) $x+5$ (l) $-(x+4)$
 (m) $\frac{1}{2}$ (n) $\frac{3x^2(3x-1)}{5x-2}$

(5) (a) $x-3$ (b) 1 (c) x (d) $\frac{(x+2)(x-3)}{x^2}$ (e) $\frac{4}{3}$ (f) x (g) $7x-1$ (h) $3x$ (i) 1

(6) (a) $\frac{2x}{3}$ (b) 4 (c) 2 (d) 5 (e) 0 (f) $x+5$ (g) $3x-5$ (h) $\frac{3x-4}{x-1}$ (i) $\frac{x+4}{x+5}$

(7) (a) $\frac{6x-7}{6}$ (b) $\frac{x+90}{45x}$ (c) $\frac{41x-54}{30}$ (d) $\frac{x+2}{x+1}$ (e) $\frac{x+2}{x-2}$ (f) $\frac{1}{x-2}$ (g) $\frac{24}{x-8}$ (h) $\frac{14}{(x-5)^2}$

(i) $\frac{11}{x^2-4}$ (j) $\frac{9+x}{1-x^2}$ (k) $\frac{x-2}{x+4}$ (l) $\frac{1}{x+6}$ (m) $\frac{1}{x-3}$ (n) 0

(8) (a) $x-1$ (b) $\frac{7}{15}$ (c) $\frac{x+4}{x-2}$ (d) $x-2$ (e) $\frac{x+4}{x-2}$ (f) $\frac{1}{7x}$ (g) $\frac{1}{x}$ (h) $\frac{1}{x}$ (i) $\frac{x-3}{x+1}$

(j) $\frac{2(a-3)}{a+6}$ (k) $\frac{x^2}{x^2-a^2}$ (l) 2 (m) 2 (n) $\frac{x-5}{x+5}$