Dawson College: Calculus II: 201-203-DW-S03: Fall 2009

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Test 1

This test is graded out of 4 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formula:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(4 marks)* Integrate the following indefinite integral:

$$\int 2\frac{1}{\sqrt[3]{x}} + 3\frac{1}{x} + 4\cos x - 3e^x\, dx = \int 2x^{-\frac{1}{3}} + \frac{3}{x} + 4\cos x - 3e^x dx$$

$$= \frac{2x^{2/3}}{2/3} + 3\ln|x| + 4\sin x - 3e^x + C$$

$$= 3x^{2/3} + 3\ln|x| + 4\sin x - 3e^x + C$$

Question 2. *(5 marks)* Evaluate the definite integral using first principles *(i.e. limit process)*:

$$\int_0^2 -x^2 + 2x - 1\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$f(x) = -x^2 + 2x - 1$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = 0 + i\frac{2}{n}$

$= \frac{2i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[-\left(\frac{2i}{n}\right)^2 + 2\left(\frac{2i}{n}\right) - 1\right]\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\frac{-4i^2}{n^2} + \frac{4i}{n} - 1\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{-4}{n^2}\sum_{i=1}^{n} i^2 + \frac{4}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{2}{\cancel{n}}\left[\frac{-4}{n^2}\,\frac{\cancel{n}(n+1)(2n+1)}{6} + \frac{\overset{2}{\cancel{4}}}{n}\,\frac{\cancel{n}(n+1)}{\cancel{2}} - \cancel{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{-\overset{4}{\cancel{8}}(n+1)(2n+1)}{\cancel{6}\,3n^2} + \frac{4(n+1)}{n} - 2\right]$$

$$= \lim_{n\to\infty}\left[\frac{-8n^2 - 12n - 4}{3n^2} + \frac{4n+4}{n} - 2\right] = \frac{-8}{3} + 4 - 2$$

$$= -\frac{2}{3}$$

Question 3. *(2 marks)* Integrate the following indefinite integral:

$$\int (y-1)(2y+1)\,dy \quad = \int 2y^2 - y - 1\,dy$$

$$= \frac{2y^3}{3} - \frac{y^2}{2} - y + C$$

Question 4. *(3 marks)* Integrate the following indefinite integral:

$$\int \frac{(t+1)(t-1)}{\sqrt{t}}\,dt \quad = \int \frac{t^2-1}{\sqrt{t}}\,dt$$

$$= \int \frac{t^2-1}{t^{1/2}}\,dt$$

$$= \int \frac{t^2}{t^{1/2}} - \frac{1}{t^{1/2}}\,dt$$

$$= \int t^{3/2} - t^{-\frac{1}{2}}\,dt$$

$$= \frac{2t^{5/2}}{5} - 2t^{1/2} + C$$

in

Question 5. *(5 marks)* Evaluate the following definite integral:

$$\int x^2\sqrt{x-1}\,dx \overset{①,②}{=} \int x^2\sqrt{u}\,du$$

$u \overset{①}{=} x-1$

$du \overset{②}{=} dx$

$u+1 = x$

$(u+1)^2 = x^2$

$u^2+2u+1 \overset{③}{=} x^2$

$$\overset{③}{=} \int (u^2+2u+1)u^{\frac{1}{2}}\,du$$

$$= \int u^{5/2} + 2u^{3/2} + u^{1/2}\,du$$

$$= \frac{2u^{7/2}}{7} + \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3} + C$$

$$\overset{①}{=} \frac{2(x-1)^{7/2}}{7} + \frac{4(x-1)^{5/2}}{5} + \frac{2(x-1)^{3/2}}{3} + C$$

Question 6. *(5 marks)* Integrate the following definite integral:

$$\int_0^1 \frac{x^2+2x}{x^3+3x^2-10}\,dx \overset{①}{=} \int_0^1 \frac{x^2+2x}{u}\,dx$$

$u \overset{①}{=} x^3 + 3x^2 - 10$

$du = (3x^2+6x)dx$

$du = 3(x^2+2x)\,dx$

$dx \overset{②}{=} \dfrac{du}{3(x^2+2x)}$

$u(0) = 0^3 + 3(0)^2 - 10 = -10$

$u(1) = 1^3 + 3(1)^2 - 10 = -6$

$$\overset{②}{=} \int_{-10}^{-6} \frac{\cancel{x^2+2x}}{u}\left(\frac{du}{3(\cancel{x^2+2x})}\right)$$

$$= \frac{1}{3}\int_{-10}^{-6} \frac{1}{u}\,du$$

$$= \frac{1}{3}\Big[\ln|u|\Big]_{-10}^{-6}$$

$$= \frac{1}{3}\Big[\big[\ln|-6|\big] - \big[\ln|-10|\big]\Big]$$

$$= \frac{1}{3}\big[\ln 6 - \ln 10\big]$$

$$= \ln\sqrt[3]{\frac{6}{10}} = \ln\sqrt[3]{\frac{3}{5}}$$

Question 7. *(5 marks)* Integrate the following definite integral:

$$\int_0^{\frac{\pi}{3}} e^{\cos 3x} \sin 3x \, dx$$

1

$u = \cos 3x$

$du = -\sin 3x \cdot 3\, dx$

$dx \overset{2}{=} \dfrac{du}{-3\sin 3x}$

$u(0) = \cos 0 = 1$

$u(\frac{\pi}{3}) = \cos \pi = -1$

$\overset{1}{=} \int_0^{\frac{\pi}{3}} e^{u} \sin 3x \, dx$

$\overset{2}{=} \int_1^{-1} e^{u} \cancel{\sin 3x} \dfrac{du}{-3\cancel{\sin 3x}}$

$= \dfrac{-1}{3} \int_1^{-1} e^{u} du$

$= \dfrac{-1}{3} \left[e^{u} \right]_1^{-1}$

$= \dfrac{-1}{3} \left[[e^{-1}] - e \right]$

$= \dfrac{e - e^{-1}}{3}$

Question 8. *(5 marks)* Evaluate the following definite integral:

$$\int_1^2 \frac{(x^2+x)}{(2x^3+3x^2)^2} dx$$

1

$u = 2x^3 + 3x^2$

$du = (6x^2 + 6x)dx$

$dx \overset{2}{=} \dfrac{du}{6(x^2+x)}$

$u(1) = 2(1)^3 + 3(1)^2 = 5$

$u(2) = 2(2)^3 + 3(2)^2 = 16 + 12$

$= 28$

$\overset{1}{=} \int_1^2 \dfrac{x^2+x}{u^2} dx$

$\overset{2}{=} \int_5^{28} \dfrac{\cancel{x^2+x}}{u^2} \dfrac{du}{6\cancel{(x^2+x)}}$

$= \dfrac{1}{6} \int_5^{28} \dfrac{1}{u^2} du$

$= \dfrac{1}{6} \int_5^{28} u^{-2} du$

$= \dfrac{1}{6} \left[\dfrac{u^{-1}}{-1} \right]_5^{28}$

$= \dfrac{1}{30} - \dfrac{1}{168} = \dfrac{23}{840}$

Question 9. *(4 marks)* Find the average value of the function $f(x) = x\sqrt{1-x^2}$ over the interval $[0,\frac{1}{2}]$.

Average of the function $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$$= \frac{1}{\frac{1}{2}-0}\int_0^{\frac{1}{2}} x\sqrt{1-x^2}\,dx$$

(1) $u = 1-x^2$

$du = -2x\,dx$

(2) $dx = \frac{du}{-2x}$

$u(0) = 1$

$u(\frac{1}{2}) = 1-\frac{1}{4} = \frac{3}{4}$

$$\overset{1,2}{=} 2\int_1^{\frac{3}{4}} x\sqrt{u}\,\frac{du}{-2x}$$

$$= -\int_1^{3/4} \sqrt{u}\,du = -\left[\frac{2u^{3/2}}{3}\right]_1^{3/4}$$

$$= -\frac{2}{3}\left(\frac{3}{4}\right)^{3/2} + \frac{2}{3}(1)^{3/2} = -\frac{\sqrt{3}}{4} + \frac{2}{3}$$

Question 10. *(5 marks)* The Fixies bike company manufactures bikes for hipsters. There are so many hipsters buying bikes that they have developed the marginal cost function for production of their most popular bike. The marginal cost function is

$$C'(x) = 0.003x^2 - 0.2x + 500$$

measured in dollars/unit and x denotes the number of units produced. Fixies has determined that the fixed cost associated to producing their most popular bike is \$100 per day. Find the total cost of producing 20 bikes per day.

$$C(x) = \int C'(x)\,dx = \int 0.003x^2 - 0.2x + 500\,dx$$

$$= 0.001x^3 - 0.1x^2 + 500x + C$$

$C(0) = 100$

$100 = 0.001(0)^3 - 0.1(0)^2 + 500(0) + C$

$100 = C$

$\therefore\ C(x) = 0.001x^3 - 0.1x^2 + 500x + 100$

$\therefore\ C(20) = 0.001(20)^3 - 0.1(20)^2 + 500(20) + 100$

$= 10\,068$

$\therefore$ the cost of produceing 20 bikes per day is 10 068 \$

Bonus Question. *(3 marks)*
Given the following basic rule of integration

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C,$$

determine the following indefinite integral

$$\int \frac{e^x}{1+e^{2x}}\,dx. \overset{①,②}{=} \int \frac{\cancel{e^x}}{1+u^2}\,\frac{du}{\cancel{e^x}}$$

$u = e^x$

$du = e^x dx$ $\quad = \arctan u + C$

$\frac{du}{e^x} = dx$ $\quad \overset{①}{=} \arctan e^x + C$