Dawson College: **Calculus II**: 201-203-DW-S03: Fall 2009

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Test 3

This test is graded out of 41 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Integrate the following indefinite integral *(using the Table of Integrals)*:

$$\int \frac{3e^x}{2+3e^{\frac{1}{2}x}}dx = 3\int \frac{e^x}{2+3e^{\frac{1}{2}x}}dx \qquad u \overset{①}{=} e^{\frac{1}{2}x}$$

$$du = \frac{e^{\frac{1}{2}x}}{2}dx$$

$$dx \overset{②}{=} \frac{2}{e^{\frac{1}{2}x}}du$$

$$= 3\int \frac{e^{x\,\frac{1}{2}x}}{2+3u}\,\frac{2}{e^{\frac{1}{2}x}}\,du$$

$$= 6\int \frac{e^{\frac{1}{2}x}}{2+3u}du$$

$$= 6\int \frac{u}{2+3u}du$$

Using integral ①

$$\int \frac{u}{a+bu}du = \frac{1}{b^2}(a+bu - a\ln|a+bu|) + C$$

where $a=2$ and $b=3$

$$= 6\left[\frac{1}{3^2}(2+3u - 2\ln|2+3u|)\right] + C$$

$$= \frac{6}{9}(2+ u - 2\ln|2+3u|) + C$$

$$\overset{①}{=} \frac{2}{3}(2+e^{\frac{1}{2}x} - 2\ln|2+e^{\frac{1}{2}x}|) + C$$

Question 2. *(5 marks)* Evaluate the following definite integral *(using the Table of Integrals)*, simplify completely:

$$\int_3^5 \frac{1}{x^2\sqrt{x^2-9}}\,dx$$

$$\int \frac{1}{x^2\sqrt{x^2-3^2}}\,dx \overset{\substack{a=3 \\ u=x}}{=} \frac{\sqrt{x^2-3^2}}{3^2x}+C$$

using integral 17

$$\int_3^5 \frac{1}{x^2\sqrt{x^2-9}}\,dx = \left[\frac{\sqrt{x^2-9}}{9x}\right]_3^5$$

$$= \frac{\sqrt{5^2-9}}{9(5)} - \frac{\sqrt{3^2-9}}{9(3)}$$

$$= \frac{\sqrt{25-9}}{45} - \frac{0}{27}$$

$$= \frac{\sqrt{16}}{45}$$

$$= \frac{4}{45}$$

Question 3. Evaluate the following limits:

a. *(3 marks)*

$$\lim_{x\to 0} \frac{\sin x}{1-\cos x}$$

I.F. $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\cos x \to 1}{\sin x \to 0} \text{ by } \hat{H}$$

D.N.E.

b. *(5 marks)*

$$\lim_{x\to 1} \frac{3e^{1-x}+2\ln x-3x}{x^2-1}$$

I.F. $\frac{0}{0}$

$$= \lim_{x\to 1} \frac{-3e^{1-x}+\frac{2}{x}-3}{2x} \quad \text{by } \hat{H}$$

$$= \frac{-3e^{1-1}+\frac{2}{1}-3}{2}$$

$$= \frac{-3+2-3}{2} = \frac{-4}{2} = -2$$

Question 4. *(5 marks)* Evaluate the following improper integral:

$$\int_1^{\infty} \frac{1}{(1+2x)^{7/2}}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{(1-2x)^{7/2}}\,dx$$

$u = 1+2x$

$du = +2dx$

$\frac{du}{+2} = dx$

$u(1) = 1+2(1) = 3$

$u(b) = 1+2b$

$$= \lim_{b\to\infty} \int_3^{1+2b} \frac{1}{u^{7/2}} \frac{du}{+2}$$

$$= \frac{+1}{2} \lim_{b\to\infty} \int_3^{1+2b} u^{-\frac{7}{2}}\,du$$

$$= \frac{1}{2} \lim_{b\to\infty} \left[\frac{-2u^{-\frac{5}{2}}}{5} \right]_3^{1+2b}$$

$$= \frac{1}{2} \lim_{b\to\infty} \left[\cancel{\frac{-2}{5(1+2b)^{5/2}}}^{\,0} + \frac{2}{5\ 3^{5/2}} \right]$$

$$= \frac{1}{\cancel{2}} \left[\frac{\cancel{2}}{5\ 3^{5/2}} \right]$$

$$= \frac{1}{5\ 3^{5/2}} = \frac{1}{5\sqrt{243}}$$

Question 5. *(3 marks)* Evaluate the following improper integral:

$$\int_{-\infty}^{-1} \frac{101}{x}\,dx = \lim_{a\to-\infty} \int_a^{-1} \frac{101}{x}\,dx$$

$$= 101 \lim_{a\to-\infty} \int_a^{-1} \frac{1}{x}\,dx$$

$$= 101 \lim_{a\to-\infty} \Big[\ln|x| \Big]_a^{-1}$$

$$= 101 \lim_{a\to-\infty} \Big[\ln|-1| - \ln|a| \Big]$$

$$= -101 \lim_{a\to\infty} \ln|a|$$

diverges to $-\infty$

Question 6. *(10 marks)* Use the trapezoidal rule with $n = 4$ to approximate the integral and compare to the exact value of the integral.

$$\int_1^2 x\ln x\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{8}\left[1\ln 1 + \cancel{2}\left(\frac{5}{\cancel{4}2}\right)\ln\frac{5}{4} + \cancel{2}\left(\frac{\cancel{6}3}{\cancel{4}}\right)\ln\frac{6}{4} + \cancel{2}\left(\frac{7}{\cancel{4}2}\right)\ln\frac{7}{4} + 2\ln 2\right]$$

$$= \frac{1}{8}\left[0 + \frac{5}{2}\ln\frac{5}{4} + 3\ln\frac{3}{2} + \frac{7}{2}\ln\frac{7}{4} + 2\ln 2\right]$$

$$\doteq 0.63990$$

$\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{4} = \dfrac{1}{4}$

$x_i = a + i\Delta x$

$x_i = 1 + i\frac{1}{4}$

$x_0 = 1$

$x_1 = \frac{5}{4}$

$x_2 = \frac{6}{4}$

$x_3 = \frac{7}{4}$

$x_4 = 2$

$$\int_1^2 x\ln x\,dx$$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$

$$= \Big[uv\Big]_1^2 - \int_1^2 v\,du$$

$$= \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\frac{1}{x}\,dx$$

$$= \left[\frac{2^2}{2}\ln 2 - \frac{1^2}{2}\ln 1\right] - \left[\frac{x^2}{4}\right]_1^2$$

$$= 2\ln 2 - \left[\frac{2^2}{4} - \frac{1}{4}\right]$$

$$= 2\ln 2 - 1 + \frac{1}{4}$$

$$= \ln 4 - 1 + \frac{1}{4}$$

$$\doteq 0.63629$$

Question 7. *(5 marks)* Verify that

$$y'' - 3y' + 2y = e^x$$

is the general solution of the differential equation

$$y = C_1 e^x + C_2 e^{2x} - xe^x.$$

Then find the particular solution that satisfy the initial condition $y(0) = 0$ and $y'(0) = 0$.

$y = C_1e^x + C_2e^{2x} - xe^x$

$y' = C_1e^x + 2C_2e^{2x} - [e^x + xe^x]$

$y' = C_1e^x + 2C_2e^{2x} - e^x - xe^x$

$y'' = C_1e^x + 4C_2e^{2x} - e^x - [xe^x + e^x]$

$y'' = C_1e^x + 4C_2e^{2x} - 2e^x - xe^{2x}$

$LHS = y'' - 3y' + 2y$

$= C_1e^x + 4C_2e^{2x} - 2e^x - xe^{2x} - 3[C_1e^x + 2C_2e^{2x} - e^x - xe^x] + 2[C_1e^x + C_2e^{2x} - xe^x]$

$= \cancel{C_1e^x} - \cancel{3C_1e^x} + \cancel{2C_1e^x} + \cancel{4C_2e^{2x}} - \cancel{6C_2e^{2x}} + \cancel{2C_2e^{2x}} - \cancel{2e^x} + 3e^x - \cancel{xe^{2x}} + \cancel{3xe^x} - \cancel{2xe^x}$

$= e^x$

$\therefore$ LHS = RHS

$\therefore$ a solution of DE

$y(0) = C_1e^0 + C_2e^{2(0)} - 0e^0$

$0 = C_1 + C_2$

$-C_2 = C_1 \quad (1)$

$y'(0) = C_1e^0 + 2C_2e^{2(0)} - e^0 - 0e^0$

$0 = C_1 + 2C_2 - 1 \quad (2)$

sub (2) into (1)

$0 = -C_2 + 2C_2 - 1$

$1 = C_2$

$\therefore C_1 = -1$

$\therefore y = -e^x + e^{2x} - xe^x$

Bonus Question. *(3 marks)* Evaluate the following improper integral.

$$\int_1^{\infty} xe^{-x}\,dx$$

$$\int_1^{\infty} x e^{-x}\,dx = \lim_{b\to\infty} \int_1^b x e^{-x}\,dx$$

$u = x$ $\quad dv = e^{-x}dx$

$du = dx$ $\quad v = -e^{-x}$

$$= \lim_{b\to\infty} \left[[uv]_1^b - \int_1^b v\,du \right]$$

$$= \lim_{b\to\infty} \left[[x(-e^{-x})]_1^b - \int_1^b -e^{-x}\,dx \right]$$

$$= \lim_{b\to\infty} \left[-be^{-b} + e^{-1} - [e^{-x}]_1^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{-b}{e^b} + e^{-1} - \cancel{e^{-b}}^{\,0} + e^{-1} \right]$$

$$= 2e^{-1} + \lim_{b\to\infty} \frac{-b}{e^b}$$ I.F. $\frac{\infty}{\infty}$

$$= 2e^{-1} + \lim_{b\to\infty} \frac{-1}{\cancel{e^b}}^{\,0}$$ by $\hat{H}$

$$= \frac{2}{e}$$