Dawson College: Calculus II: 201-203-DW-S03: Fall 2009

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Test 4

This test is graded out of 44 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Find the general solution of the following differential equation.

$$y' = \frac{y^2xe^{x^2}}{1+y^2}$$

$$\frac{dy}{dx} = \frac{y^2xe^{x^2}}{1+y^2}$$

$$\frac{1+y^2}{y^2}dy = xe^{x^2}dx$$

$$\int \frac{1+y^2}{y^2}dy = \int xe^{x^2}dx$$

$$\int \frac{1}{y^2} + \frac{y^2}{y^2}dy = \int xe^{x^2}dx$$

$u = x^2$

$du = 2xdx$

$\frac{du}{2x} = dx$

$$\int \frac{1}{y^2} + 1\,dy = \int xe^u \frac{du}{2x}$$

$$\frac{-1}{y} + y = \frac{1}{2}\int e^u du$$

$$\frac{-1}{y} + y = \frac{1}{2}e^u + C$$

$$y - \frac{1}{y} = \frac{1}{2}e^{x^2} + C$$

Question 2. *(5 marks)* Find the solution to the following following initial value problem:

$$y' = 5xy - 3x$$

where $y(0) = 1$.

$$\frac{dy}{dx} = 5xy - 3x$$

$$\frac{dy}{dx} = x(5y-3)$$

$$\frac{dy}{(5y-3)} = xdx$$

$$\int \frac{1}{5y-3}dy = \int xdx$$

$$\frac{1}{5}\ln|5y-3| = \frac{x^2}{2} + C$$

$$\ln|5y-3| = \frac{5x^2}{2} + C$$

$$e^{\ln|5y-3|} = e^{\left(\frac{5x^2}{2}+C\right)}$$

$$5y - 3 = e^C e^{\frac{5x^2}{2}} \qquad \text{let } A = e^C$$

$$5y - 3 = Ae^{\frac{5x^2}{2}}$$

$$5y = Ae^{\frac{5x^2}{2}} + 3$$

$$y = \frac{A}{5}e^{\frac{5x^2}{2}} + \frac{3}{5}$$

$y(0) = 1$

$$1 = \frac{A}{5}e^0 + \frac{3}{5}$$

$$\frac{2}{5} = \frac{A}{5}$$

$$2 = A$$

$$\therefore\ y = \frac{2}{5}e^{\frac{5x^2}{2}} + \frac{3}{5}$$

(5marks)

Question 3. Find the third Taylor Polynomial of $f(x) = xe^{-3x} + \cos 2x$ at $x = 0$

$f(x) = xe^{-3x} + \cos 2x$

$f(0) = 0 + 1$
$= 1$

$f'(x) = e^{-3x} + xe^{-3x}(-3) - 2\sin 2x$
$= e^{-3x} - 3xe^{-3x} - 2\sin 2x$

$f'(0) = 1 - 0 - 0 = 1$

$f''(x) = -3e^{-3x} - 3\left[e^{-3x} - 3xe^{-3x}\right] - 4\cos 2x$
$= -6e^{-3x} + 9xe^{-3x} - 4\cos 2x$

$f''(0) = -6 + 0 - 4 = -10$

$f'''(x) = 18e^{-3x} + 9\left[e^{-3x} - 3xe^{-3x}\right] + 8\sin 2x$
$= 18e^{-3x} + 9e^{-3x} - 27xe^{-3x} + 8\sin 2x$

$f'''(0) = 18 + 9 - 0 + 0 = 27$

$\therefore P_3(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$= 1 + x - \frac{10x^2}{2!} + \frac{27x^3}{3!}$

$= 1 + x - 5x^2 + \frac{9}{2}x^3$

Question 4. Determine the convergence or divergence of the following sequences.

a. *(2 marks)*

$$a_n = \frac{\ln n}{n}$$

$\lim_{n\to\infty} \frac{\ln n}{n}$ I.F. $\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$ by $\hat{H}$

$= \lim_{n\to\infty} \frac{1}{n} = 0$

b. *(2 marks)*

$$a_n = \frac{2n^2 - n - 1}{3n^2 + 3n + 101}$$

$\lim_{n\to\infty} \frac{(2n^2 - n - 1)\left(\frac{1}{n^2}\right)}{(3n^2 + 3n + 101)\left(\frac{1}{n^2}\right)}$

$= \lim_{n\to\infty} \frac{\frac{2n^2}{n^2} - \frac{n}{n^2} \to 0 - \frac{1}{n^2} \to 0}{\frac{3n^2}{n^2} + \frac{3n}{n^2} \to 0 + \frac{101}{n^2} \to 0}$

$= \frac{2}{3}$

Question 5. *(5 marks)* Determine if the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

partial fractions:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

let $n=0$

$$1 = A(0+1) + B(0)$$

$$1 = A$$

let $n=-1$

$$1 = A(-1+1) + B(-1)$$

$$-1 = B$$

$$\therefore \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1} + a_n$$

$$= \left[\frac{1}{1} - \cancel{\frac{1}{2}}\right] + \left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \frac{1}{4}\right] + \dots + \left[\frac{1}{n-2} - \cancel{\frac{1}{n-1}}\right] + \left[\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right] + \left[\cancel{\frac{1}{n}} - \frac{1}{n+1}\right]$$

$$= 1 - \frac{1}{n+1}$$

$$S = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \left[1 - \overbrace{\cancel{\frac{1}{n+1}}}^{0}\right]$$

$$= 1$$

Question 6. *(5 marks)* Determine if the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{5^n+1}{7^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{5^n}{7^{n+1}} + \frac{1}{7^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{5^n}{7^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{7^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{5^n}{77^n} + a_0 - a_0 + \sum_{n=1}^{\infty} \frac{1}{77^n} + b_0 - b_0$$

$$= \sum_{n=0}^{\infty} \frac{1}{7}\left(\frac{5}{7}\right)^n - a_0 + \sum_{n=0}^{\infty} \frac{1}{7}\left(\frac{1}{7}\right)^n - b_0$$

where $a_n = \frac{1}{7}\left(\frac{5}{7}\right)^n$ and $b_n = \frac{1}{7}\left(\frac{1}{7}\right)^n$

$$= \frac{1/7}{1-5/7} - \frac{1}{7} + \frac{1/7}{1-1/7} - \frac{1}{7}$$

$$= \frac{1/7}{2/7} - \frac{1}{7} + \frac{1/7}{6/7} - \frac{1}{7}$$

$$= \frac{1}{2} + \frac{1}{6} - \frac{2}{7}$$

$$= \frac{8}{21}$$

Question 7. *(5 marks)* Determine if the following series converges or diverges, justify your answer and state the test used.

$$\sum_{n=1}^{\infty} \frac{e^{-\ln n}}{n}$$

lets use the integral test.

let $f(x) = \frac{e^{-\ln x}}{x}$

- $f(x)$ positive for $x \geq 1$? ✓
- $f(x)$ continuous for $x \geq 1$? ✓
- $f(x)$ decreasing for $x \geq 1$?

$$f'(x) = \frac{-\frac{e^{-\ln x}}{x} x - e^{-\ln x}}{x^2}$$

$$= \frac{-2e^{-\ln x}}{x^2} < 0$$

$$\int_1^{\infty} \frac{e^{-\ln x}}{x}\,dx$$

$$= \lim_{b \to \infty} \int_1^b \frac{e^{-\ln x}}{x}\,dx$$

$u = -\ln x$ $\quad$ $u(1) = -\ln 1 = 0$

$du = \frac{-dx}{x}$ $\quad$ $u(b) = -\ln b$

$-x\,du = dx$

$$= \lim_{b \to \infty} \int_0^{-\ln b} \frac{e^u}{x} - x\,du$$

$$= -\lim_{b \to \infty} \left[e^u \right]_0^{-\ln b}$$

$$= -\lim_{b \to \infty} \left[e^{-\ln b} - e^0 \right]$$

$$= 1$$

$\therefore \sum_{n=1}^{\infty} \frac{e^{-\ln n}}{n}$ converges by the integral test

Question 8. *(5 marks)* Determine if the following series converges or diverges, justify your answer and state the test used.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^2+1}}$$

$$\lim_{n \to \infty} \frac{n}{\sqrt{2n^2+1}} = \lim_{n \to \infty} \sqrt{\frac{n^2}{2n^2+1}}$$

$$= \lim_{n \to \infty} \sqrt{\frac{\cancel{n^2}}{\cancel{n^2}(2+\frac{1}{n^2})}}$$

$$= \frac{1}{\sqrt{2}} \neq 0$$

$\therefore$ by the n^{th} term divergence test the series diverges

Question 9. *(5 marks)* Determine If the following series converges or diverges, justify your answer and state the test used.

$$\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^5+1}}$$

Lets use the comparison test.

$$a_n = \frac{n}{\sqrt{n^5+1}} \le \frac{n}{\sqrt{n^5}} = \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} = b_n$$

Note that $\sum_{n=1}^{\infty} b_n$ converges since p-series where $p = \frac{3}{2} > 1$

$\therefore$ by comparison test converges since a larger series converges.

Bonus. *(3 marks)* Determine If the following series converges or diverges, justify your answer and state the test used.

$$\sum_{n=0}^{\infty} \frac{n}{e^n} = \sum_{n=0}^{\infty} n e^{-n}$$

Lets use integral test

Let $f(x) = xe^{-x}$

- $f(x)$ is positive for $x \ge 0$?
- $f(x)$ is continuous for $x \ge 0$?
- $f(x)$ is decreasing for $x \ge 0$?

$$f'(x) = e^{-x} - xe^{-x}$$
$$= e^{-x}(1-x) < 0$$

for $x > 1$

$$\int_0^{\infty} xe^{-x}\,dx$$

$$= \lim_{b\to\infty} \int_0^b xe^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[[uv]_0^b - \int_0^b v\,du \right]$$

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$= \lim_{b\to\infty} \left[[x \cdot -e^{-x}]_0^b - \int_0^b -e^{-x}\,dx \right]$$

$$= \lim_{b\to\infty} \left[-be^{-b} + 0e^{0} + [-e^{-x}]_0^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{-b}{e^b} - \cancel{e^{-b}} + e^0 \right]$$

$I.F \ \frac{-\infty}{\infty}$

$$= 1 - \lim_{b\to\infty} \frac{-1}{e^b}$$

$$= 1$$

$\therefore \sum_{n=0}^{\infty} \frac{n}{e^n}$ converges by the integral test.