Dawson College: Calculus II: 201-NYB-05-S02: Fall 2009

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Test 1

This test is graded out of 47 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formula:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(4 marks)* Integrate the following indefinite integral:

$$\int 2\frac{1}{\sqrt[3]{x}} + 3\csc x - \frac{3}{\sqrt{3-x^2}}\,dx = \int 2x^{-\frac{1}{3}} + 3\csc x - \frac{3}{\sqrt{(\sqrt{3})^2 - x^2}}$$

$$= \frac{6x^{2/3}}{2} - 3\ln|\csc x + \cot x| - 3\arcsin\frac{x}{\sqrt{3}} + C$$

Question 2. *(5 marks)* Evaluate the definite integral using first principles *(i.e. limit process)*:

$$\int_0^2 -x^2 + 2x - 1\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$f(x) = -x^2 + 2x - 1$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\Delta x$
$= \frac{2i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[-\left(\frac{2i}{n}\right)^2 + 2\left(\frac{2i}{n}\right) - 1\right]\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[-\frac{4i^2}{n^2} + \frac{4i}{n} - 1\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{-4}{n^2}\sum_{i=1}^{n} i^2 + \frac{4}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{-\cancel{8}^{4}}{n^{\cancel{3}2}}\left[\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,3}\right] + \frac{\cancel{8}^{4}}{n^{\cancel{2}}}\left[\frac{\cancel{n}(n+1)}{\cancel{2}}\right] - \frac{2\cancel{n}}{\cancel{n}}\right]$$

$$= \lim_{n\to\infty} \frac{-8n^2 - 12n - 4}{3n^2} + \frac{4n+4}{n} - 2$$

$$= \frac{-8}{3} + 4 - 2$$

$$= \frac{-2}{3}$$

Question 3. *(3 marks)* Integrate the following indefinite integral:

$$\int \frac{e^{\ln y^2}}{y}\,dy = \int \frac{y^2}{y}\,dy$$

$$= \int y\,dy$$

$$= \frac{y^2}{2} + C$$

Question 4. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{z^3 - z^2 + z - 1}{z^2 + 5}\,dz = \int z - 1 + \frac{-4z + 4}{z^2+5}\,dz$$

$$\begin{array}{r} z - 1 \\ z^2 + 0z + 5 \,\overline{)\, z^3 - z^2 + z - 1} \\ -(z^3 + 0z^2 + 5z) \\ \hline -z^2 - 4z - 1 \\ -(-z^2 - 0z - 5) \\ \hline -4z + 4 \end{array}$$

use substitution

$$= \int z - 1\,dz - 4\int \frac{z}{z^2+5}\,dz + 4\int \frac{1}{z^2+5}\,dz$$

$$= \frac{z^2}{2} - z - 2\ln|z^2+5| + \frac{4}{\sqrt{5}}\arctan\frac{z}{\sqrt{5}} + C$$

Question 5. *(5 marks)* Evaluate the following definite integral:

$$\int_1^2 (x^2+x)(2x^3+3x^2)^2\,dx \overset{①,②}{=} \int_5^{28} \cancel{(x^2+x)}\,u^2 \frac{du}{6\cancel{(x^2+x)}}$$

$u \overset{①}{=} 2x^3+3x^2$

$du = (6x^2+6x)dx$

$dx \overset{②}{=} \dfrac{du}{6(x^2+x)}$

$u(1) = 2(1)^3+3(1)^2 = 5$

$u(2) = 2(2)^3+3(2)^2 = 28$

$$= \frac{1}{6}\int_5^{28} u^2\,du$$

$$= \frac{1}{6}\left[\frac{u^3}{3}\right]_5^{28}$$

$$= \frac{28^3}{18} - \frac{5^3}{18} = \frac{28^3-5^3}{18} = \frac{21827}{18}$$

Question 6. *(5 marks)* Differentiate the following expression:

$$\frac{d}{dx}\left[\int_x^{\arctan x} f(t)\,dt\right]$$

$\| \; Ⓐ$

$$\int_x^{\arctan x} f(t)dt = \int_x^0 f(t)dt + \int_0^{\arctan x} f(t)dt$$

$$= -\int_0^x f(t)dt + \int_0^{\arctan x} f(t)dt \quad Ⓐ$$

$$\frac{d}{dx}\left[-\int_0^x f(t)dt + \int_0^{\arctan x} f(t)dt\right]$$

$$= -\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] + \frac{d}{dx}[h(x)]$$

by 2^{nd} FTC

$$= -f(x) + K'(g(x))\,g'(x)$$

$$= -f(x) + f(\arctan x)\cdot\frac{1}{1+x^2}$$

Let $h(x) = \displaystyle\int_0^{\arctan x} f(t)dt$

$h(x) = K(g(x))$ where $K(x) = \displaystyle\int_0^x f(t)dt$

$K'(x) = f(x)$ by 2^{nd} FTC

and $g(x) = \arctan x$

$g'(x) = \dfrac{1}{1+x^2}$

Question 7. *(4 marks)* Find the average of the function $f(x) = \tan x$ on the interval $[-\frac{\pi}{6}, \frac{\pi}{6}]$.

Average value of $f(x)$ $= \frac{1}{b-a} \int_a^b f(x)dx$

$= \frac{1}{\frac{\pi}{6} - (\frac{-\pi}{6})} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan x \, dx$

$= \frac{3}{\pi} \cdot 0$ since $f(x)$ is odd

$= 0$

Lets show that $f(x) = \tan x$ is odd

$f(-x) = \tan(-x) = \frac{\sin(-x)}{\cos(-x)}$

$= \frac{-\sin x}{\cos x}$ since $\sin x$ is odd

$= -\tan x$

$= -f(x)$

Question 8. Given that $f(-x) = -f(x)$, $g(-x) = g(x)$, $\int_0^a f(x)\,dx = 3$ and $\int_0^a g(x)\,dx = 5$, evaluate the following definite integrals:

a. *(2 marks)*

$\int_a^a 2g(x)\,dx = 0$ since bounds are the same

b. *(2 marks)*

$\int_{-a}^a 3f(x)\,dx = 3 \int_{-a}^{a} f(x) = 3 \cdot 0$ since $f(x)$ is odd

$= 0$

c. *(2 marks)*

$\int_{-a}^a -2g(x)\,dx = -2 \int_{-a}^{a} g(x) = -2 \cdot 2 \int_0^a g(x)dx$ since $g(x)$ is even

$= -4 \cdot 5$

$= -20$

Question 9. *(5 marks)* Evaluate the following definite integral:

$$\int_{\frac{\pi}{4}}^{0} \frac{\sec^2 x}{1+\tan^2 x}\,dx \overset{①,②}{=} \int_1^0 \frac{\cancel{\sec^2 x}}{1+u^2}\frac{du}{\cancel{\sec^2 x}} = \Big[\arctan u\Big]_1^0$$

$u \overset{①}{=} \tan x$

$du = \sec^2 x\,dx$

$dx \overset{②}{=} \dfrac{du}{\sec^2 x}$

$u(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

$u(0) = \tan 0 = 0$

$= \arctan 0 - \arctan 1$

$= 0 - \frac{\pi}{4}$

$= \frac{-\pi}{4}$

Or as Fatima cleverly observed $\quad 1+\tan^2 x = \sec^2 x$

Hence $\displaystyle\int_{\frac{\pi}{4}}^{0} \frac{\cancel{\sec^2 x}}{\cancel{\sec^2 x}}\,dx = \int_{\frac{\pi}{4}}^{0} 1\,dx = \Big[x\Big]_{\frac{\pi}{4}}^{0} = 0 - \frac{\pi}{4} = \frac{-\pi}{4}$

Question 10. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{\sec(\arcsin 2x)}{\sqrt{1-4x^2}}\,dx \overset{①,②}{=} \int \frac{\sec(u)}{\cancel{\sqrt{1-(2x)^2}}}\,\frac{\cancel{\sqrt{1-(2x)^2}}\,du}{2}$$

$u \overset{①}{=} \arcsin 2x$

$du = \dfrac{1}{\sqrt{1-(2x)^2}} \cdot 2\,dx$

$dx \overset{②}{=} \dfrac{\sqrt{1-(2x)^2}\,du}{2}$

$= \frac{1}{2}\int \sec u\,du$

$= \frac{1}{2}\ln|\sec u + \tan u| + C$

$\overset{①}{=} \frac{1}{2}\ln|\sec \arcsin 2x + \tan \arcsin 2x| + C$

Bonus Question.

a. *(1 mark)* State the First Fundamental Theorem of Calculus.

b. *(1 mark)* State the Mean Value Theorem.

c. *(3 marks)* Prove the First Fundamental Theorem of Calculus.

See notes