Dawson College: **Calculus II**: 201-NYB-05-S02: Fall 2009

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Test 2

This test is graded out of 43 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(3 marks)* Integrate the following indefinite integral:

$\int \cos^2 2\theta \, d\theta$ use half-angle identity. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$= \int \frac{1+\cos 4\theta}{2} \, d\theta$

$= \frac{\theta}{2} + \frac{\sin 4\theta}{8} + C$

Question 2. *(5 marks)* Evaluate the following definite integral:

$\int_0^{\frac{1}{2\sqrt{2}}} \arcsin(2x)\, dx = \left[uv\right]_0^{\frac{1}{2\sqrt{2}}} - \int_0^{\frac{1}{2\sqrt{2}}} v\,du$

where $u = \arcsin 2x$

$du = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2\,dx$

$dv = dx$

$v = x$

$= \left[x \arcsin 2x\right]_0^{\frac{1}{2\sqrt{2}}} - \int_0^{\frac{1}{2\sqrt{2}}} \frac{2x}{\sqrt{1-(2x)^2}}\,dx$

$u = 1-4x^2$

$du = -8x\,dx$

$\frac{du}{-8x} = dx$

$u(0) = 1$

$u\left(\frac{1}{2\sqrt{2}}\right) = 1 - 4\left(\frac{1}{2\sqrt{2}}\right)^2$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

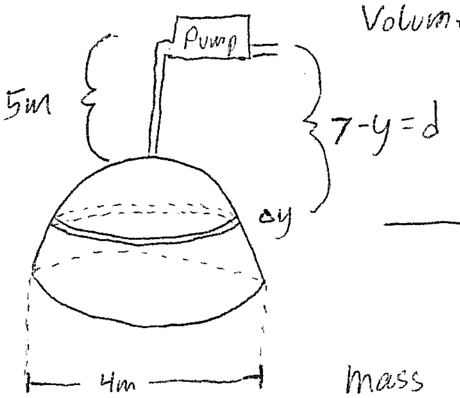
$= \left[\frac{1}{2\sqrt{2}} \arcsin \frac{1}{\sqrt{2}}\right] - \left[0 \arcsin 0\right] - \int_1^{\frac{1}{2}} \frac{2x}{\sqrt{u}} \frac{du}{-8x}$

$= \frac{1}{2\sqrt{2}}\left(\frac{\pi}{4}\right) + \frac{1}{4}\left[2u^{1/2}\right]_1^{1/2}$

$= \frac{\pi}{8\sqrt{2}} + \frac{1}{4}\left[2\left(\frac{1}{2}\right)^{1/2} - 2\right]$

$= \frac{\pi}{8\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{1}{2}$

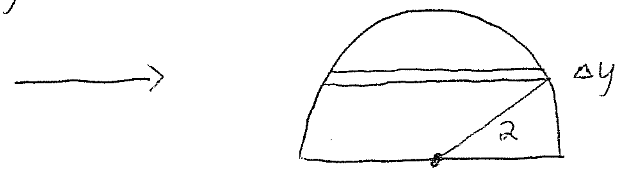
Question 3. *(5 marks)* A tank has the shape of a hemisphere with its round side up. The tank is 4 meters across the bottom. How much work is done in emptying the tank by pumping the water 5 m over the top? ($\rho = 1000\frac{kg}{m^3}$ *and* $g = 9.8\frac{m}{s^2}$)



Volume of slice: $\Delta V = \pi r^2 \Delta y$

$\Delta V = \pi x^2 \Delta y$

$\Delta V = \pi(4-y^2)\Delta y$



$x^2 + y^2 = 2^2$

$x^2 = 2^2 - y^2$

mass of slice: $\Delta m = \Delta V \rho$

$= \pi 1000 (4-y^2)\Delta y$

Force of slice: $\Delta F = \Delta m g$

$= \pi 1000 (4-y^2)(9.8)\Delta y$

$= \pi 9800 (4-y^2)\Delta y$

Work to move slice: $\Delta W = \Delta F d$

$= \pi 9800 (4-y^2)(7-y)\Delta y$

$= \pi 9800 \left[y^3 - 7y^2 - 4y + 28\right]\Delta y$

$$\therefore \text{Work} = \int_0^2 \pi 9800 \left[y^3 - 7y^2 - 4y + 28\right]dy$$

$$= 9800\pi \left[\frac{y^4}{4} - \frac{7y^3}{3} - \frac{4y^2}{2} + 28y\right]_0^2$$

$$= 9800\pi \left[\frac{2^4}{4} - \frac{7 \cdot 2^3}{3} - \frac{4 \cdot 2^2}{2} + 28(2)\right]$$

$$= \frac{980\,000}{3}\pi \ N \cdot m$$

Question 4. *(5 marks)* Find the arclength of the graph of the function

$$y = \ln(\sin x)$$

over the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$.

$$y' = \frac{1}{\sin x} \cos x$$

$$y' = \frac{\cos x}{\sin x}$$

$$y' = \cot x$$

$$S = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{1 + (y')^2}\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{1 + \cot^2 x}\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\csc^2 x}\, dx \quad \text{using identity}$$

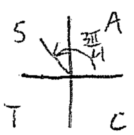
$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc x\, dx$$

$$= \Big[-\ln|\csc x + \cot x|\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \left[-\ln\left|\csc\tfrac{3\pi}{4} + \cot\tfrac{3\pi}{4}\right|\right] + \ln\left|\csc\tfrac{\pi}{4} + \cot\tfrac{\pi}{4}\right|$$

<u>note:</u>

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \csc\frac{\pi}{4} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = 1 \Rightarrow \cot\frac{\pi}{4} = 1$$



$$\csc\frac{3\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \csc\frac{3\pi}{4} = \sqrt{2}$$

$$\tan\frac{3\pi}{4} = -1 \Rightarrow \cot\frac{3\pi}{4} = -1$$

$$= -\ln|\sqrt{2} - 1| + \ln|\sqrt{2} + 1|$$

$$= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Question 5. *(5 marks)* Sketch the region bounded by the graphs of the functions

$$f(x) = \sqrt{x-1} + 2 \text{ and } g(x) = x + 1$$

and find the area of the region.

Lets find the intersection between $f(x)$ and $g(x)$:

$$f(x) = g(x)$$

$$\sqrt{x-1} + 2 = x + 1$$

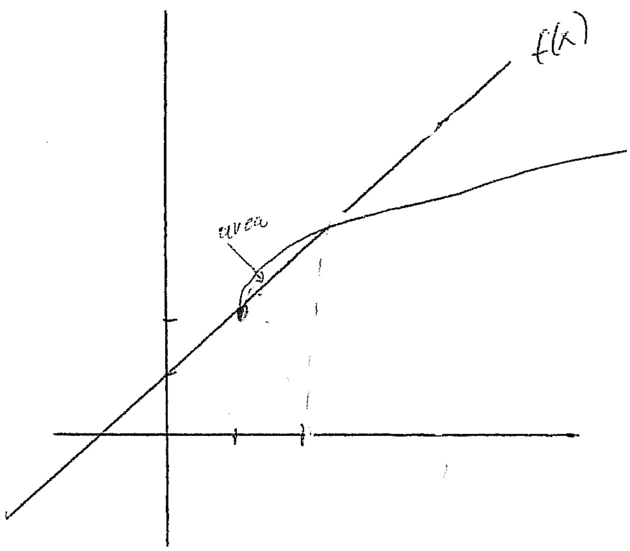
$$(\sqrt{x-1})^2 = (x-1)^2$$

$$x - 1 = x^2 - 2x + 1$$

$$0 = x^2 - 3x + 2$$

$$0 = (x-2)(x-1)$$

$$x = 2 \qquad x = 1$$



$$\text{area} = \int_1^2 \text{Top curve} - \text{bottom curve}\, dx$$

$$= \int_1^2 \sqrt{x-1} + 2 - (x+1)\, dx$$

$$= \int_1^2 \sqrt{x-1} - x + 1\, dx$$

$$= \left[\frac{2(x-1)^{3/2}}{3} - \frac{x^2}{2} + x \right]_1^2$$

$$= \left[\frac{2(2-1)^{3/2}}{3} - \frac{\cancel{2}^2}{\cancel{2}} + 2 \right] - \left[\cancel{\frac{2(1-1)^{3/2}}{3}}^{\,0} - \frac{1^2}{2} + 1 \right]$$

$$= \left[2/3 - 2 + 2 \right] + \frac{1}{2} - 1$$

$$= 1/6$$

Question 6. *(5 marks)* Set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region defined by:

$f(x) = x^2 - 4x + 3$, $g(x) = -x^2 + 2x + 3$

about $x = -1$.

Lets find the intersection of the two curve.

$$f(x) = g(x)$$
$$x^2 - 4x + 3 = -x^2 + 2x + 3$$
$$2x^2 - 6x = 0$$
$$x(2x - 6) = 0$$
$$x = 0 \qquad x = 3$$

Lets sketch $g(x)$

y-int: $(0, 3)$

x-int: $0 = -x^2 + 2x + 3$
$$0 = x^2 - 2x - 3$$
$$0 = (x-3)(x+1)$$
$$x = 3 \qquad x = -1$$

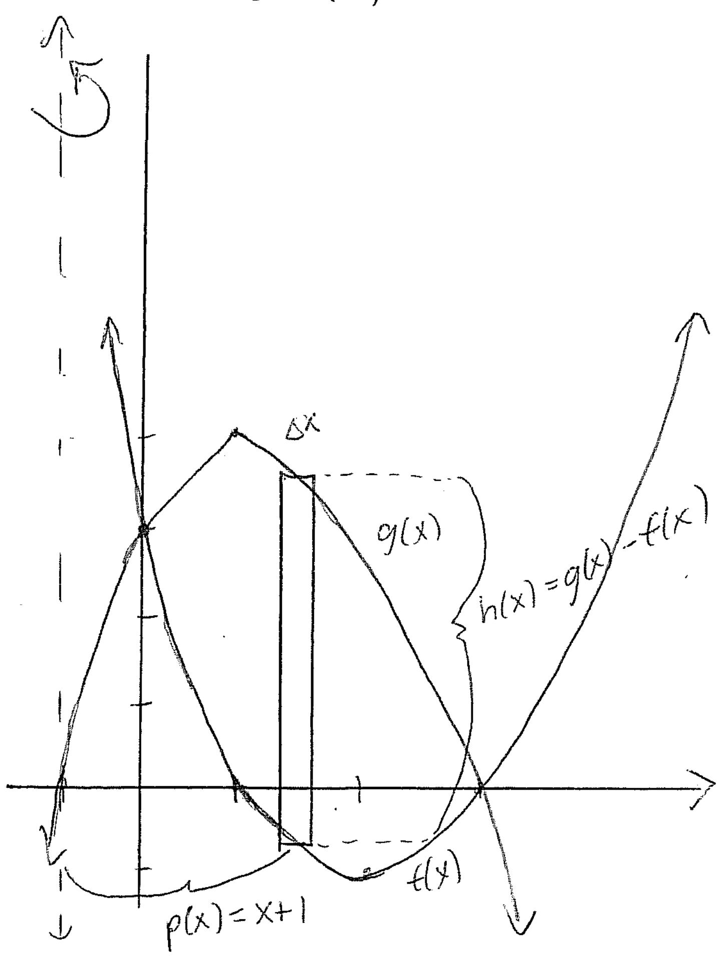
vertex: $g(x) = -x^2 + 2x + 3$
$$= -[x^2 - 2x - 3]$$
$$= -[(x^2 - 2x + 1) - 1 - 3]$$
$$= -[(x-1)^2 - 4]$$
$$= -(x-1)^2 + 4$$
$$\therefore (1, +4)$$

Lets sketch $f(x)$:

y-int: $(0, 3)$

x-int: $0 = x^2 - 4x + 3$
$$0 = (x-3)(x-1)$$
$$x = 3 \qquad x = 1$$

vertex: $f(x) = x^2 - 4x + 3$
$$= (x^2 - 4x + 4) - 4 + 3$$
$$= (x-2)^2 - 1$$
$$\therefore (2, 1)$$



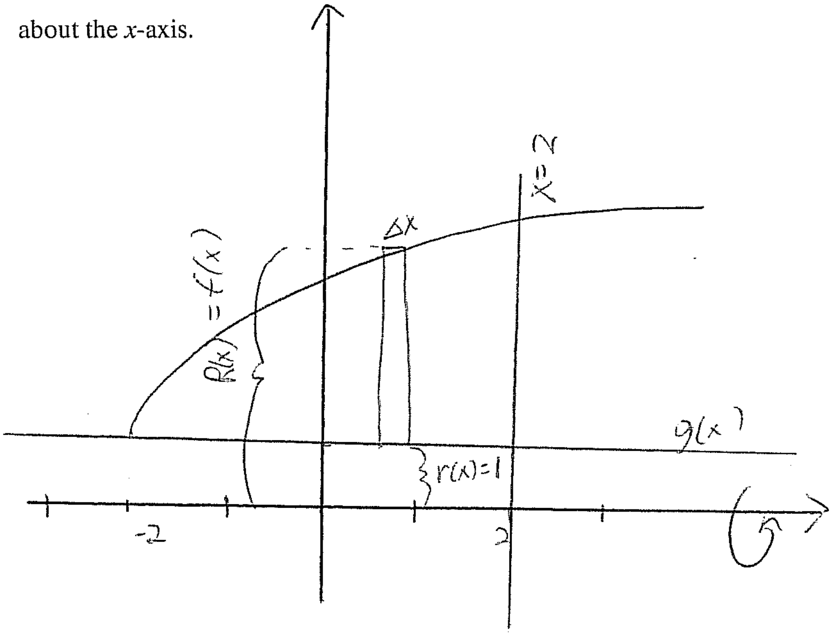
$$\Delta V = 2\pi p(x) h(x) \Delta x$$
$$= 2\pi (x+1)(g(x) - f(x))\Delta x$$
$$= 2\pi (x+1)(-x^2 + 2x + 3 - x^2 + 4x - 3)\Delta x$$
$$= 2\pi (x+1)(-2x^2 + 6x)\Delta x$$

$$V = \int_0^3 2\pi (x+1)(-2x^2 + 6x)\,dx$$
$$= 2\pi \int_0^3 -2x^3 + 6x^2 - 2x^2 + 6x\,dx$$
$$= 2\pi \int_0^3 -2x^3 + 4x^2 + 6x\,dx$$
$$= 2\pi \left[\frac{-2x^4}{4} + \frac{4x^3}{3} + \frac{6x^2}{2} \right]_0^3$$
$$= 2\pi \left[\frac{-2 \cdot 3^4}{4} + \frac{4(3)^3}{3} + \frac{6(3)^2}{2} \right] = 45\pi$$

Question 7. *(5 marks)* Set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region defined by:

$$f(x) = \sqrt{x+2}+1,\ g(x) = 1,\ x = 2$$

about the x-axis.



$$\Delta V = \pi\left[(R(x))^2 - (r(x))^2\right]\Delta x$$

$$= \pi\left[\left(\sqrt{x+2}+1\right)^2 - 1^2\right]\Delta x$$

$$= \pi\left[x+2+2\sqrt{x+2}+1-1\right]\Delta x$$

$$V = \int_{-2}^{2} \pi\left[x+2+2\sqrt{x+2}\right]dx$$

$$= \left[\pi\left(\frac{x^2}{2} + 2x + \frac{4(x+2)^{3/2}}{3}\right)\right]_{-2}^{2}$$

$$= \pi\left(\frac{2^2}{2} + 2(2) + \frac{4(2+2)^{3/2}}{3}\right) - \pi\left(\frac{(-2)^2}{2} + 2(-2) + \frac{4(-2+2)^{3/2}}{3}\right)$$

$$= \pi\left(6 + \frac{4}{3}(8)\right) - \pi(2-4)$$

$$= \frac{56\pi}{3}$$

Question 8. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int \frac{4x^2+2x+1}{x^2(x+1)}\,dx$$

$$\frac{4x^2+2x-1}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1}$$

$$4x^2+2x-1 = (Ax+B)(x+1) + Cx^2$$

Let $x=0$

$$-1 = (A(0)+B)(0+1)+C0^2$$

$$-1 = B$$

Let $x=-1$

$$4(-1)^2+2(-1)-1 = (A(-1)+B)(\underbrace{\cancel{-1+1}}_{})^{0} + C(-1)^2$$

$$1 = C$$

Let $x=1$

$$4(1)^2+2(1)-1 = (A(1)+B)(1+1)+C(1)^2$$

$$5 = (2A+2(-1)) + 1$$

$$2+4 = 2A$$

$$3 = A$$

$$\therefore \int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int \frac{3x-1}{x^2} + \frac{1}{x+1}\,dx$$

$$= \int \frac{3\cancel{x}}{x^{\cancel{2}}} - \frac{1}{x^2} + \frac{1}{x+1}\,dx$$

$$= 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$

Question 9.*(5 marks)* Integrate the following indefinite integral:

$$\int x^3\sqrt{2+x^2}\,dx$$

$$x \overset{(1)}{=} \sqrt{2}\tan\theta \Rightarrow \frac{x}{\sqrt{2}} = \tan\theta$$
$$dx \overset{(2)}{=} \sqrt{2}\sec^2\theta\,d\theta$$

$$\overset{(1),(2)}{=} \int (\sqrt{2}\tan\theta)^3\sqrt{2+(\sqrt{2}\tan\theta)^2}\;\sqrt{2}\sec^2\theta\,d\theta$$

$$= \int (\sqrt{2})^3\tan^3\theta\sqrt{2(1+\tan^2\theta)}\;\sqrt{2}\sec^2\theta\,d\theta$$

$$= \int \sqrt{8}\tan^3\theta\sqrt{2\sec^2\theta}\;\sqrt{2}\sec^2\theta\,d\theta$$

$$= 4\sqrt{2}\int \tan^3\theta\sec^3\theta\,d\theta$$

$$= 4\sqrt{2}\int \tan^2\theta\sec^2\theta\tan\theta\sec\theta\,d\theta$$

$$= 4\sqrt{2}\int (\sec^2\theta-1)\sec^2\theta\tan\theta\sec\theta\,d\theta$$

$$= 4\sqrt{2}\int (\sec^4\theta-\sec^2\theta)\tan\theta\sec\theta\,d\theta$$

$$u \overset{(3)}{=} \sec\theta$$
$$du \overset{(4)}{=} \sec\theta\tan\theta\,d\theta \Rightarrow d\theta = \frac{du}{\sec\theta\tan\theta}$$

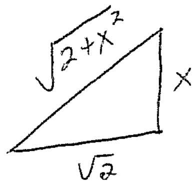
$$\overset{(3),(4)}{=} 4\sqrt{2}\int (u^4-u^2)\cancel{\tan\theta\sec\theta}\,\frac{du}{\cancel{\sec\theta\tan\theta}}$$

$$= 4\sqrt{2}\int u^4-u^2\,du$$

$$= 4\sqrt{2}\left[\frac{u^5}{5}-\frac{u^3}{3}\right]+C$$

$$\overset{(3)}{=} 4\sqrt{2}\left[\frac{\sec^5\theta}{5}-\frac{\sec^3\theta}{3}\right]+C$$

$$\tan\theta = \frac{x}{\sqrt{2}} = \frac{opp}{adj}$$



$$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{2+x^2}}{\sqrt{2}}$$

$$= \frac{4\sqrt{2}\left(\frac{\sqrt{2+x^2}}{\sqrt{2}}\right)^5}{5} - \frac{4\sqrt{2}\left(\frac{\sqrt{2+x^2}}{\sqrt{2}}\right)^3}{3} + C$$

$$= \frac{\cancel{4\sqrt{2}}\left(\sqrt{2+x^2}\right)^5}{5\cdot\cancel{4\sqrt{2}}} - \frac{\overset{2}{\cancel{4\sqrt{2}}}\left(\sqrt{2+x^2}\right)^3}{\cancel{2\sqrt{2}}\,3} + C$$

$$= \frac{\left(\sqrt{2+x^2}\right)^5}{5} - 2\,\frac{\left(\sqrt{2+x^2}\right)^3}{3} + C$$

Bonus Question. *(3 marks)* Integrate the following indefinite integral:

$$\int x \text{ arcsin} x \, dx$$

$u = \arcsin x$ $\quad$ $dv = x^2 dx$

$du = \dfrac{1}{\sqrt{1-x^2}} dx$ $\quad$ $v = \dfrac{x^2}{2}$

$$= uv - \int v \, du$$

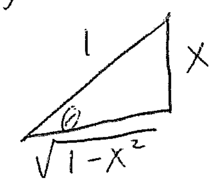
① $x = \sin\theta$

② $dx = \cos\theta \, d\theta$

using ①

$\theta = \arcsin x$

using ①



$\therefore \cos\theta = \sqrt{1-x^2}$

$$= \frac{x^2}{2}\arcsin x - \int \frac{x^2}{2\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2}\arcsin x - \int \frac{\sin^2\theta}{2\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cancel{\sqrt{\cos^2\theta}}} \cancel{\cos\theta} \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1 - \cos 2\theta \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{\arcsin\theta}{4} + \frac{1 \cancel{2}}{\cancel{4} 2}\sin\theta\cos\theta + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin\theta + \frac{1}{2}x\sqrt{1-x^2} + C$$