Dawson College: **Calculus II**: 201-NYB-05-S02: Fall 2009

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Test 3

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Evaluate the following limit.

a. *(5 marks)*

$$\lim_{x\to\infty} x\arcsin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} x\arcsin\left(\frac{1}{x}\right) \quad \text{I.F. } \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\arcsin\left(\frac{1}{x}\right)}{\frac{1}{x}} \quad \text{I.F. } \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(\frac{\cancel{-1}}{\cancel{x^2}}\right)}{\frac{\cancel{-1}}{\cancel{x^2}}} \quad \text{by } \hat{H}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} = 1$$

b. *(5 marks)*

$$\lim_{x\to\infty}(1+x)^{1/x}$$

$$y = \lim_{x\to\infty} (1+x)^{1/x}$$

$$\ln y = \lim_{x\to\infty} \ln(1+x)^{\frac{1}{x}}$$

$$\ln y = \lim_{x\to\infty} \frac{1}{x}\ln(1+x) \quad \text{I.F. } \frac{\infty}{\infty}$$

$$\ln y = \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} \quad \text{by } \hat{H}$$

$$\ln y = \lim_{x\to\infty} \frac{1}{1+x}$$

$$\ln y = 0$$

$$y = e^0$$

$$y = 1$$

Question 2. Integrate the following improper integral.

a. *(5 marks)*

$$\int_1^7 \frac{2}{(1-x)^{5/3}}dx = \lim_{a\to 1} \int_a^7 \frac{2}{(1-x)^{5/3}} dx = \lim_{a\to 1^+} \int_{1-a}^{-6} \frac{2}{u^{5/3}} -du$$

$u = 1-x$

$du = -dx$

$-du = dx$

$u(a) = 1-a$

$u(7) = 1-7 = -6$

$$= -2 \lim_{a\to 1^+} \int_{1-a}^{-6} u^{-5/3} du$$

$$= \cancel{-2} \lim_{a\to 1^+} \left[\frac{+3u^{-2/3}}{\cancel{2}} \right]_{1-a}^{-6}$$

$$= 3(-6)^{-2/3} - \lim_{a\to 1^+} \cancel{\frac{3}{(1-a)^{2/3}}}^{\infty}$$

$\therefore$ integral diverges.

b. *(5 marks)*

$$\int_1^\infty \frac{\arctan x}{1+x^2} dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{\arctan x}{x^2+1} dx$$

$u = \arctan x$

$du = \frac{1}{1+x^2} dx$

$dx = (1+x^2) du$

$u(1) = \arctan 1 = \frac{\pi}{4}$

$u(b) = \arctan b$

$$= \lim_{b\to\infty} \int_{\pi/4}^{\arctan b} \frac{u \cancel{(x^2+1)}}{\cancel{(x^2+1)}} du$$

$$= \lim_{b\to\infty} \frac{u^2}{2}$$

$$= \lim_{b\to\infty} \frac{(\arctan b)^2}{2} - \frac{(\pi/4)^2}{2}$$

$$= \frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{(\pi/4)^2}{2}$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{32}$$

$$= \frac{3\pi^2}{32}$$

Question 3. Find the sum of the following series if it converges, if it diverges justify.

a. *(5 marks)*

$$\sum_{n=1}^{\infty} \frac{3^{n+1}+2^{n}}{7^{n+1}}$$

b. *(5 marks)*

$$\sum_{n=10}^{\infty} \frac{2\ {}^{3} - \ \ +1}{3\ {}^{3}+2\ {}^{2}-1}$$

$$\lim_{n\to\infty} \frac{2n^3 - n + 1 \left(\frac{1}{n^3}\right)}{3n^3+2n^2-1\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{\frac{2n^3}{n^3} - \frac{n}{n^3} + \frac{1}{n^3}}{\frac{3n^3}{n^3} + \frac{2n^2}{n^3} - \frac{1}{n^3}}$$

$$= \frac{2}{3}$$

$\therefore$ by the n^{th} term divergence test the series diverges

a)
$$= \sum_{n=1}^{\infty} \frac{3^{n+1}}{7^{n+1}} + \sum_{n=1}^{\infty} \frac{2^{n}}{7^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{3\,3^n}{7\,7^n} + \sum_{n=1}^{\infty} \frac{2^n}{7\,7^n}$$

$$= \sum_{n=1}^{\infty} \frac{3}{7}\left(\frac{3}{7}\right)^n + a_0 - a_0 + \sum_{n=1}^{\infty} \frac{1}{7}\left(\frac{2}{7}\right)^n + b_0 - b_0$$

where $a_n = \frac{3}{7}\left(\frac{3}{7}\right)^n$ and $b_n = \frac{1}{7}\left(\frac{2}{7}\right)^n$

$$= \sum_{n=0}^{\infty} \frac{3}{7}\left(\frac{3}{7}\right)^n - \frac{3}{7} + \sum_{n=0}^{\infty} \frac{1}{7}\left(\frac{2}{7}\right)^n - \frac{1}{7}$$

$$= \frac{\frac{3}{7}}{1-3/7} - \frac{3}{7} + \frac{1/7}{1-2/7} - \frac{1}{7}$$

$$= \frac{3/7}{4/7} - \frac{3}{7} + \frac{1/7}{5/7} - \frac{1}{7}$$

$$= \frac{3}{4} - \frac{4}{7} + \frac{1}{5}$$

$$= \frac{53}{140}$$

Question 4. Determine the convergence or divergence of the series.

a. *(5 marks)*

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n! 3^n}$$

Lets use the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 2^{n+1}}{(n+1)!\, 3^{n+1}} \cdot \frac{n!\, 3^n}{(-1)^n 2^n} \right|$$

$$= \lim_{n\to\infty} \frac{2 \cancel{2^n} \cancel{n!} \cancel{3^n}}{(n+1) \cancel{n!}\, 3 \cancel{3^n} \cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2}{3(n+1)} = 0 < 1$$

$\therefore$ series converges by ratio test.

b. *(5 marks)*

$$\sum_{n=1}^{\infty} \frac{e^{-\ln x}}{n}$$

Lets use the integral test

let $f(x) = \dfrac{e^{-\ln x}}{x}$

- $f(x)$ is positive for $x \geq 1$
- $f(x)$ is continuous for $x \geq 1$
- $f(x)$ is decreasing for $x \geq 1$?

$$f'(x) = \frac{e^{-\ln x} \frac{-1}{\cancel{x}} \cancel{x} - e^{-\ln x}}{x^2} = \frac{-2e^{-\ln x}}{x^2} < 0 \text{ for } x \geq 1.$$

$$\int_1^{\infty} \frac{e^{-\ln x}}{x}\, dx = \lim_{b\to\infty} \int_1^b \frac{e^{-\ln x}}{x}\, dx$$

$u = -\ln x$

$du = \frac{-1}{x} dx$

$-x\, du = dx$

$u(1) = -\ln 1 = 0$

$u(b) = -\ln b$

$$= \lim_{b\to\infty} \int_0^{-\ln b} \frac{e^u}{\cancel{x}} \cancel{-x}\, du$$

$$= -\lim_{b\to\infty} \left[e^u \right]_0^{-\ln b}$$

$$= -\lim_{b\to\infty} \left[\cancel{e^{-\ln b}}^{\,0} - e^0 \right]$$

$$= 1$$

$\therefore$ converges by integral test

Question 5. *(5 marks)* Determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^5 - n^2 + 1}}$$

Lets use the limit comparison test. Let $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5}} = \sum_{n=1}^{\infty} b_n$ be the test series. The test series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conveges since it is a p-series where $p = 3/2 > 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{\sqrt{n^5-n^2+1}} \cdot \frac{\sqrt{n^5}}{n} = \lim_{n\to\infty} \cancel{\frac{n+1}{n}}^{\,1} \lim_{n\to\infty} \frac{\sqrt{n^5}}{\sqrt{n^5-n^2+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^5}{n^5-n^2+1}} = 1$$

Question 6. *(5 marks)* Find the 4^{th} degree Taylor polynomial of the function $f(x) = x^2e^{2-2x} + \cos(2-2x)$ centered at $x = 1$.

$$f(x) = x^2e^{2-2x} + \cos(2-2x)$$

$$f'(x) = 2xe^{2-2x} + x^2e^{2-2x}(-2) - \sin(2-2x)(-2)$$

$$= 2xe^{2-2x} - 2x^2e^{2-2x} + 2\sin(2-2x)$$

$$f''(x) = 2e^{2-2x} + 2xe^{2-2x}(-2) - 4xe^{2-2x} - 2x^2e^{2-2x}(-2) + 2\cos(2-2x)(-2)$$

$$= 2e^{2-2x} - 8xe^{2-2x} + 4x^2e^{2-2x} - 4\cos(2-2x)$$

$$f(1) = 1+1 = 2$$

$$f'(1) = 2-2+0 = 0$$

$$f''(1) = 2-8+4-4 = -6$$

$$P_2(x) = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!}$$

$$= 2 + \frac{0(x-1)}{1!} - \frac{6(x-1)^2}{2}$$

$$= 2 - 3(x-1)^2$$

Bonus Question. Let $f(x) = e^{3-3x}$

a. *(1 mark)* Find a_n the n^{th} term of the Taylor polynomial of the function $f(x)$ centered at $x = 1$.

b. *(2 marks)* For which value of x does the Taylor series

$$\sum_{n=0}^{\infty} a_n$$

converge.

a) $f(x) = e^{3-3x}$ $\quad f(1) = 1$

$f'(x) = -3e^{3-3x}$ $\quad f'(1) = -3$

$f''(x) = 9e^{3-3x}$ $\quad f''(1) = 9$

$f'''(x) = -27e^{3-3x}$ $\quad f'''(1) = -27$

$f^{(IV)}(x) = 81e^{3-3x}$ $\quad f^{(IV)}(1) = 81$

$\vdots \qquad \vdots$

$f^{(n)}(x) = (-3)^n e^{3-3x}$ $\quad f^{(n)}(1) = (-3)^n$

$$\therefore \quad a_n = \frac{(-3)^n}{n!}(x-1)^n$$

b) Let's use the ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1}(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-3)^n(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-3)\cancel{(-3)^n}(x-1)\cancel{(x-1)^n}\cancel{n!}}{(n+1)\cancel{(n!)}\cancel{(-3)^n}\cancel{(x-1)^n}} \right|$$

$$= \lim_{n\to\infty} \frac{3(x-1)}{n+1}$$

$$= 0 < 1$$

$\therefore$ converges for all x values.