Dawson College: **Linear Algrbra**: 201-105-DW-S04: Fall 2009

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Test 1

This test is graded out of 49 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(10 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcrcl} 3x_1 & - & x_2 & + & x_3 & - & 3x_4 & + & x_5 & = & 3 \\ 4x_1 & + & 3x_2 & - & x_3 & + & x_4 & + & x_5 & = & 5 \\ 11x_1 & + & 5x_2 & - & x_3 & & & + & 3x_5 & = & 2 \end{array}$$

$$\begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 4 & 3 & -1 & 1 & 1 & 5 \\ 11 & 5 & -1 & 0 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{array}{l} \\ 3R_2 \\ 3R_3 \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 12 & 9 & -3 & 3 & 3 & 15 \\ 33 & 15 & -3 & 0 & 9 & 6 \end{bmatrix}$$

$$\sim \begin{array}{l} \\ -4R_1+R_2 \to R_2 \\ -11R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 0 & 13 & -7 & 15 & -1 & 3 \\ 0 & 26 & -14 & 33 & -2 & -27 \end{bmatrix}$$

$$\sim \begin{array}{l} \\ \\ -2R_2+R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 0 & 13 & -7 & 15 & -1 & 3 \\ 0 & 0 & 0 & 3 & 0 & -33 \end{bmatrix}$$

$$\sim \begin{array}{l} \\ \\ \frac{1}{3}R_3 \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 0 & 13 & -7 & 15 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 & -11 \end{bmatrix}$$

$$\sim \begin{array}{l} 3R_3+R_1 \to R_1 \\ -15R_3+R_2 \to R_2 \\ \end{array} \begin{bmatrix} 3 & -1 & 1 & 0 & 1 & -30 \\ 0 & 13 & -7 & 0 & -1 & 168 \\ 0 & 0 & 0 & 1 & 0 & -11 \end{bmatrix}$$

$$\sim \begin{array}{l} 13R_1 \\ \\ \end{array} \begin{bmatrix} 39 & -13 & 13 & 0 & 13 & -390 \\ 0 & 13 & -7 & 0 & -1 & 168 \\ 0 & 0 & 0 & 1 & 0 & -11 \end{bmatrix}$$

$$\sim \begin{array}{l} R_1+R_2 \to R_1 \\ \\ \end{array} \begin{bmatrix} 39 & 0 & 4 & 0 & 12 & -222 \\ 0 & 13 & -7 & 0 & -1 & 168 \\ 0 & 0 & 0 & 1 & 0 & -11 \end{bmatrix}$$

$$\sim \begin{array}{l} \frac{1}{39}R_1 \\ \frac{1}{13}R_2 \\ \end{array} \begin{bmatrix} 1 & 0 & \frac{4}{39} & 0 & \frac{12}{39} & \frac{-222}{39} \\ 0 & 1 & \frac{-7}{13} & 0 & \frac{-1}{13} & \frac{168}{13} \\ 0 & 0 & 0 & 1 & 0 & -11 \end{bmatrix}$$

The free variables are:
x_3, x_5

So,
$x_3 = s$
$x_5 = t$

$\therefore$ $x_1 = \frac{-222}{39} - \frac{4}{39}s - \frac{12}{39}t$

$x_2 = \frac{168}{13} + \frac{t}{13} + \frac{7}{13}s$

$x_3 = s$

$x_4 = -11$

$x_5 = t$

Question 2. Consider the matrices:

$$A=\begin{bmatrix}-1 & 2 & 5\\ 2 & 3 & -6\\ 1 & 0 & -2\end{bmatrix},\ B=\begin{bmatrix}3 & 2 & -1\\ 0 & -4 & 1\end{bmatrix}\ C=\begin{bmatrix}2 & -1\\ -3 & 4\\ 0 & -1\end{bmatrix}\ D=\begin{bmatrix}1 & 0\\ -2 & -2\end{bmatrix}$$

a. *(2 marks)* Compute the following, if possible.

$C-D$ not possible since not same dimension

b. *(3 marks)* Compute the following, if possible.

$BC-D$

c. *(3 marks)* Compute the following, if possible.

C^tB^t

d. *(3 marks)* Compute the following, if possible.

AB

e. *(5 marks)* Find E, if possible.

$(D^t+I-E^t)^{-1}=D$

$$BC-D$$
$$=\begin{bmatrix}3 & 2 & -1\\ 0 & -4 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ -3 & 4\\ 0 & -1\end{bmatrix}-\begin{bmatrix}1 & 0\\ -2 & -2\end{bmatrix}$$
$$=\begin{bmatrix}0 & 6\\ 12 & -17\end{bmatrix}-\begin{bmatrix}1 & 0\\ -2 & -2\end{bmatrix}=\begin{bmatrix}-1 & 6\\ 14 & -15\end{bmatrix}$$

$$C^tB^t=\begin{bmatrix}2 & -3 & 0\\ -1 & 4 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 2 & -4\\ -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 12\\ 6 & -17\end{bmatrix}$$

AB impossible, # of col. of A does not match the # of rows of B.
$3\times3\ \ 2\times3$

$$(D^t+I-E^t)^{-1}=D$$
$$\begin{bmatrix}1 & 0\\ -2 & -2\end{bmatrix}=\left(\begin{bmatrix}1 & -2\\ 0 & -2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-E^t\right)^{-1}$$
$$\begin{bmatrix}1 & 0\\ -2 & -2\end{bmatrix}^{-1}=\left[\left(\begin{bmatrix}2 & -2\\ 0 & -1\end{bmatrix}-E^t\right)^{-1}\right]^{-1}$$
$$\frac{1}{1(-2)}\begin{bmatrix}-2 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}2 & -2\\ 0 & -1\end{bmatrix}-E^t$$
$$E^t=\begin{bmatrix}2 & -2\\ 0 & -1\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 1 & \frac{1}{2}\end{bmatrix}$$
$$E^t=\begin{bmatrix}1 & -2\\ 1 & -\frac{1}{2}\end{bmatrix}$$

$$(E^t)^t=\begin{bmatrix}1 & -2\\ 1 & -\frac{1}{2}\end{bmatrix}^t$$
$$E=\begin{bmatrix}1 & 1\\ -2 & -\frac{1}{2}\end{bmatrix}$$

Question 3. *(5 marks)* Given the following augmented matrix in row-echelon form, solve the system using back substitution.

$$\begin{bmatrix} 1 & 2 & 2 & \frac{5}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \iff \begin{aligned} (5)\ & x_1 + 2x_2 + 2x_3 + \tfrac{5}{2}x_4 = \tfrac{1}{2} \\ (3)\ & x_2 - x_3 + 3x_4 = 2 \\ (1)\ & x_4 = 2 \end{aligned}$$

x_3 is a free variable, hence (2) $x_3 = t$

sub (1), (2) into (3) $\quad x_2 - t + 3(2) = 2$

(4) $x_2 = -4 + t$

sub (1), (2), (4) into (5)

$$x_1 + 2(-4+t) + 2(t) + \frac{5}{\cancel{2}}(\cancel{2}) = \frac{1}{2}$$

$$x_1 = \frac{7}{2} - 4t$$

$\therefore (x_1, x_2, x_3, x_4) = (\frac{7}{2} - 4t, -4+t, t, 2)$

Question 4. *(3 marks)* Given the following augmented matrix solve the system.

$$\begin{bmatrix} 1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow 0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

$$0 = 1$$

the system is inconsistent

$\therefore$ no solutions

Question 5. *(5 marks)* Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 4 \\ -3 & -6 & 10 \end{bmatrix}$$

$$[A \mid I]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ -3 & -6 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -2R_1 + R_2 \to R_2 \\ +3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 10 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} 3R_3 + R_1 \to R_1 \\ -10R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 10 & 0 & 3 \\ 0 & 1 & 0 & -32 & 1 & -10 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \; -2R_2 + R_1 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 74 & -2 & 23 \\ 0 & 1 & 0 & -32 & 1 & -10 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 74 & -2 & 23 \\ -32 & 1 & -10 \\ 3 & 0 & 1 \end{bmatrix}$$

Question 6. *(5 marks)* Write the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

as a product of elementary matrices.

$$\sim_{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\sim_{-2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ obtained from $-2R_1+R_2 \to R_2$ on I

$E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ obtained from $-2R_2+R_1 \to R_1$ on I

Hence

$$E_2 E_1 A = I$$
$$E_2^{-1} E_2 E_1 A = E_2^{-1} I$$
$$I E_1 A = E_2^{-1}$$
$$E_1 A = E_2^{-1}$$
$$E_1^{-1} E_1 A = E_1^{-1} E_2^{-1}$$
$$I A = E_1^{-1} E_2^{-1}$$
$$A = E_1^{-1} E_2^{-1}$$

note: $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$E_2^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Question 7. *(5 marks)* Solve the following system by inverting the coefficient matrix.

$$\begin{aligned} -2x_1 \quad\quad &= 1 \\ x_1 - x_2 &= 2 \end{aligned}$$

$Ax = b$

$$\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 \end{bmatrix}$$

$Ax = b$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

$x = A^{-1}b$

$$x = \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}$$

Bonus Question. *(3 marks)* Consider the following system:

$$\begin{aligned} 2x - y + z &= 3 \\ x + 2y - az &= 22 \\ 4x + 3y - z &= b \end{aligned} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 2 & -a & 22 \\ 4 & 3 & -1 & b \end{array}\right]$$

where $a, b \in \mathbb{R}$, determine the values of a, b so that the system has

i. a unique solution,

ii. infitely many solutions,

iii. no solutions.

$$2R_2 \begin{bmatrix} 2 & -1 & 1 & 3 \\ 2 & 4 & -2a & 44 \\ 4 & 3 & -1 & b \end{bmatrix} \sim \begin{array}{c} -R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 2 & -1 & 1 & 3 \\ 0 & 5 & -2a-1 & 41 \\ 0 & 5 & -3 & -6+b \end{bmatrix}$$

$$\sim \; -R_2+R_3 \begin{bmatrix} 2 & -1 & 1 & 3 \\ 0 & 5 & -2a-1 & 41 \\ 0 & 0 & 2a-2 & -47+b \end{bmatrix}$$

iii) no solution if $b = 47$ and $a \neq 1$

ii) infinitely many solutions if $b = 47$ and $a = 1$

i) unique solution if $a \neq 1$