Dawson College: Linear Algebra: 201-105-DW-S04: Fall 2009

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Test 2

This test is graded out of 47 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Let

$$A=\begin{bmatrix} 0&1&1&1&9&3&4&1&0&9\\ 0&0&1&1&4&9&2&7&7&9\\ 0&0&0&1&1&4&9&2&7&7\\ 0&0&0&0&1&1&4&9&2&7\\ 0&0&0&0&0&1&1&4&9&2\\ 0&0&0&0&0&0&1&1&4&9\\ 0&0&0&0&0&0&0&1&1&4\\ 0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}, C=\begin{bmatrix} a&b&c&d\\ e&f&g&h\\ i&j&k&l\\ m&n&o&p \end{bmatrix} D=\begin{bmatrix} q&r&s&t\\ u&v&w&x\\ y&z&\alpha&\beta\\ \delta&\varepsilon&\gamma&\aleph \end{bmatrix}$$

a. *(2 marks)* Is A invertible, justify.

b. *(3 marks)* If B is a 10×10 matrix show that AB is not invertible.

c. *(5 marks)* If $\det(C)=2$ and $\det(D)=-3$ then compute $\det(2C^3D^T(3CD)^{-1}(\ C)^3)$.

a) not invertible since $\det A = 0$.

b) $\det(AB) = \det A \det B$
$= 0 \det B$
$= 0$

$\therefore$ AB not invertible.

c) $\det 2C^3 \det D^T \det(3CD)^{-1} \det(\ C)^3$

$= 2^4(\det C)^3 \det D \dfrac{1}{\det 3CD} (\det C)^3$

$= 2^4(\det C)^{\overset{2}{3}} \cancel{\det D} \dfrac{1}{3^4 \cancel{\det C}\, \cancel{\det D}} (\det C)^3$

$= \dfrac{2^4(\det C)^2}{3^4} (\det C)^3$

$= \dfrac{2^4}{3^4} (\det C)^5$

$= \dfrac{2^4}{3^4} (\det C)^5$

$= \dfrac{2^4 2^5}{3^4}$

$= \dfrac{2}{3^4}$

$= \dfrac{512}{81}$

Question 2. *(5 marks)* Use Cramer's rule to solve for x_2 without solving for x_1, x_3, x_4.

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + 2x_4 &= 2 \\ x_2 - x_3 + x_4 &= 2 \\ 2x_2 - 3x_3 - 3x_4 &= 0 \\ 5x_4 &= 0 \end{aligned} \iff \begin{bmatrix} 2 & 2 & 3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -3 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = \frac{\det A_2}{\det A}$$

$$\det A = \begin{vmatrix} 2 & 2 & 3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -3 & -3 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$\underset{-2R_2+R_3 \to R_3}{\sim} \begin{vmatrix} 2 & 2 & 3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 5 \end{vmatrix} = -10$$

$$A_2 = \begin{bmatrix} 2 & 2 & 3 & 2 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad \det A_2 = -60$$

$$\therefore\ x_2 = \frac{-60}{-10} = 6$$

Question 3. *(5 marks)* If

$$A = \begin{bmatrix} 0 & 0 & 3 & 2 \\ -2 & 3 & -1 & 3 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix},$$

compute det(A) *(use cofactor expansion to compute)*

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}$$

$$= 0C_{11} + (-2)C_{21} + 0C_{31} + 0C_{41}$$

$$= -2C_{21}$$

$$= -2(-1)^{2+1} \begin{vmatrix} 0 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 2[a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}]$$

$$= 2[0C_{11} + 3C_{12} + 2C_{13}]$$

$$= 2\left[3(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}\right]$$

$$= 2[-3[6-1] + 2[4-2]]$$

$$= 2[-15 + 4]$$

$$= -22$$

Question 4. Let $S = \{1,2,3,4,5,6,7\}$.

a. *(2 marks)* Determine the parity of the permutation $(2,1,3,4,7,6,5)$ of the set S.

b. *(2 marks)* Is $(1,2,3,4,5,6,7)$ a permutation of the set S, justify.

a) # of inversions $= 1 + 0 + 0 + 0 + 2 + 1 = 4$

$\therefore$ the permutation is even

b) yes, since there is no omission and no repetition.

Question 5. *(5 marks)* Consider the matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad B = \begin{bmatrix} d & e & f \\ 3a & 3b & 3c \\ 4a+g & 4b+h & 4c+i \end{bmatrix}.$$

If $\det(A) = -13$, compute $\det(B)$.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$\sim \begin{array}{c} \\ 3R_2 \\ 4R_2+R_3 \end{array} \begin{bmatrix} d & e & f \\ 3a & 3b & 3c \\ 4a+g & 4b+h & 4c+i \end{bmatrix} = B$$

$$(-1)(3) \det A = \det B$$

$$(-1)(3) \ (-13) = \det B$$

$$39 = \det B$$

Question 6. Consider the matrix:

$$A = \begin{bmatrix} 2 & 3 & -4 & 0 \\ -1 & 5 & 2 & 1 \\ 3 & 8 & -6 & 2 \\ 0 & 2 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a. *(2 marks)* Compute $\det(A)$.

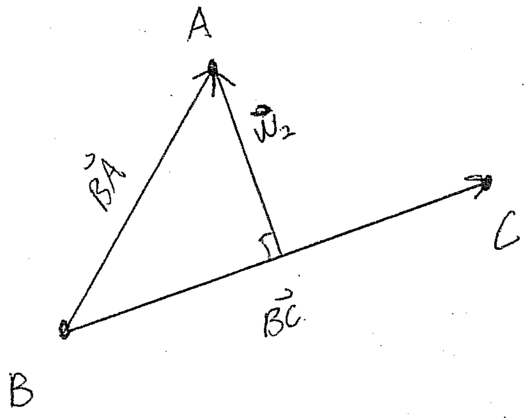
b. *(2 marks)* Compute B^{101}.

a) $\det A = 0$ since $C_3 = 2C_1$

b) $$B^{101} = \begin{bmatrix} 1^{101} & 0 \\ 0 & (-1)^{101} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Question ~~6~~ 7. *(7 marks)* Find the area *(use projections)* of the triangle defined by the vertices:

$A(2,-1,3)$, $B(-2,1,-3)$, $C(7,-5,2)$.



$$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$$

base $= \|\vec{BC}\|$

height $= \|\vec{w}_2\|$

$$\vec{BC} = C - B = (7,-5,2) - (-2,1,-3) = (9,-6,5)$$
$$\vec{BA} = A - B = (2,-1,3) - (-2,1,-3) = (4,-2,6)$$

$$\|\vec{BC}\| = \sqrt{9^2 + (-6)^2 + (5)^2} = \sqrt{142}$$

$$\vec{w}_2 = \vec{BA} - \text{proj}_{\vec{BC}}\vec{BA}$$
$$= (4,-2,6) - \frac{\vec{BA}\cdot\vec{BC}}{\vec{BC}\cdot\vec{BC}}\vec{BC}$$
$$= (4,-2,6) - \frac{4(9)+(-2)(-6)+5(6)}{9(9)+(-6)(-6)+5(5)}(9,-6,5)$$
$$= (4,-2,6) - \frac{78}{142}(9,-6,5)$$
$$= (4,-2,6) - \left(\frac{351}{71}, \frac{-234}{71}, \frac{195}{71}\right)$$
$$= \left(\frac{-67}{71}, \frac{92}{71}, \frac{231}{71}\right)$$

$$\|\vec{w}_2\| = \sqrt{\left(\frac{-67}{71}\right)^2 + \left(\frac{92}{71}\right)^2 + \left(\frac{231}{71}\right)^2}$$
$$= \sqrt{\frac{934}{71}}$$

$$\therefore \text{Area} = \frac{1}{2}\|\vec{w}_2\|\|\vec{BC}\|$$
$$= \frac{1}{2}\sqrt{\frac{934}{71}}\sqrt{142}$$
$$= \frac{1}{2}\sqrt{\frac{934(142)}{71}}$$
$$= \sqrt{467}$$
$$\doteq 21.6 \ u^2$$

Question 8. (5 marks) If

$$A = \begin{bmatrix} 2 & 1 & 2 & 4 & 6 \\ 0 & -2 & 1 & -3 & 3 \\ 0 & 2 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix},$$

compute det(A)(use elementary operations).

$$A = \begin{bmatrix} 2 & 1 & 2 & 4 & 6 \\ 0 & -2 & 1 & -3 & 3 \\ 0 & 2 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix} \sim \begin{matrix} R_2+R_3 \to R_3 \\ R_4 \leftrightarrow R_5 \end{matrix} \begin{bmatrix} 2 & 1 & 2 & 4 & 6 \\ 0 & -2 & 1 & -3 & 3 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = B$$

$$\therefore (-1)\det A = \det B$$
$$(-1)\det A = -8$$
$$\det A = 8$$

Question 9. (2 marks) Find all 3×3 diagonal matrices Y that satisfy $Y^2 - 7Y - 10I = 0$.

Using the quadratic formula

$$Y = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(-10)}}{2} I$$
$$= \frac{7 \pm \sqrt{89}}{2} I$$

Bonus Question. (3 marks) Establish the identity:

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$$

$$(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= \vec{u}\cdot(\vec{u}+\vec{v}) + \vec{v}\cdot(\vec{u}+\vec{v}) + \vec{u}\cdot(\vec{u}-\vec{v}) - \vec{v}\cdot(\vec{u}-\vec{v})$$
$$= u\cdot u + \cancel{u\cdot v} + \cancel{v\cdot u} + v\cdot v + u\cdot u - \cancel{u\cdot v} - \cancel{v\cdot u} + v\cdot v$$
$$= ||\vec{u}||^2 + ||\vec{v}||^2 + ||\vec{u}||^2 + ||\vec{v}||^2 = 2||\vec{u}||^2 + 2||\vec{v}||^2$$