Dawson College: **Linear Algebra**: 201-105-DW-S04: Fall 2009

Name: ____________________
Student ID: ____________________

Test 3

This test is graded out of 31 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(3 marks)* Find the area of the parallelogram determined by $\mathbf{u} = (2,0,-3)$ and $\mathbf{v} = (-1,1,-1)$.

$$\text{Area} = \|\vec{u} \times \vec{v}\| = \left\| \left(\begin{vmatrix} 0 & 1 \\ -3 & -1 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ -3 & -1 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \right) \right\|$$

$$\begin{matrix} 2 & -1 \\ 0 & 1 \\ -3 & -1 \end{matrix} \quad = \|(3, 5, 2)\| = \sqrt{3^2+5^2+2^2} = \sqrt{38}$$

Question 2. *(3 marks)* Compute the scalar triple product of $\mathbf{u} = (2,1,-3)$, $\mathbf{v} = (0, 1, 0)$ and $\mathbf{w} = (2,1,-2)$.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 2 & 1 & -3 \\ 0 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= 1(-1)^{2+2} \begin{vmatrix} 2 & -3 \\ 2 & -2 \end{vmatrix}$$

$$= 2$$

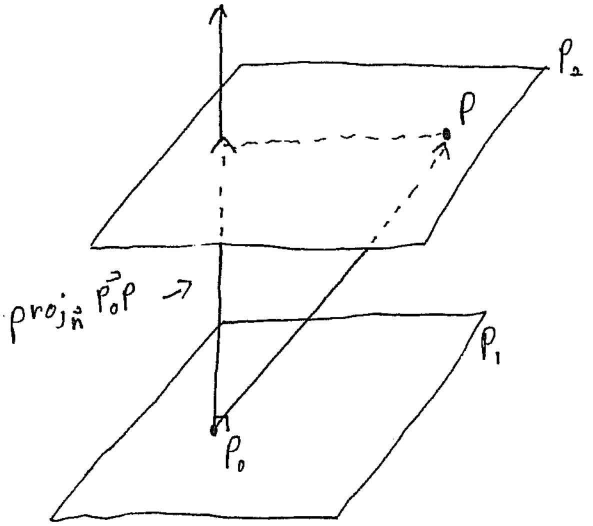
Question 3. *(3 marks)* Determine of the line $(x,y,z) = (1,0,2) + t(1,2,2)$ and the plane $2x - 3y + 5z = 0$ intersect, if so find the intersection.

So $L \begin{cases} x = 1+t \\ y = 2t \\ z = 2+2t \end{cases}$ Sub into plane to find at what value t there is intersection

$$2(1+t) - 3(2t) + 5(2+2t) = 0$$
$$2 + 2t - 6t + 10 + 10t = 0$$
$$6t = -12$$
$$t = -2$$

$\therefore$ intersection at $(x,y,z) = (1,0,2) - 2(1,2,2)$
$$= (-1,-4,-2)$$

Question 4. *(5 marks)* Find the distance between the two planes $\underbrace{-2x+y-z=10}_{P_1}$ and $\underbrace{4x-2y+2z=-4}_{P_2}$. *(using projections)*



Lets find P, let $y=0$, $z=0$, $4x-2(0)+2(0)=-4$
$x=-1$

$\therefore P(-1,0,0)$

Lets find P_0 let $y=0$, $z=0$, $-2x+0-0=10$
$x=-5$

$\therefore P_0(-5,0,0)$

$\vec{P_0P} = P - P_0 = (-1,0,0)-(-5,0,0)=(4,0,0)$

$\vec{n} = (-2,1,-1)$

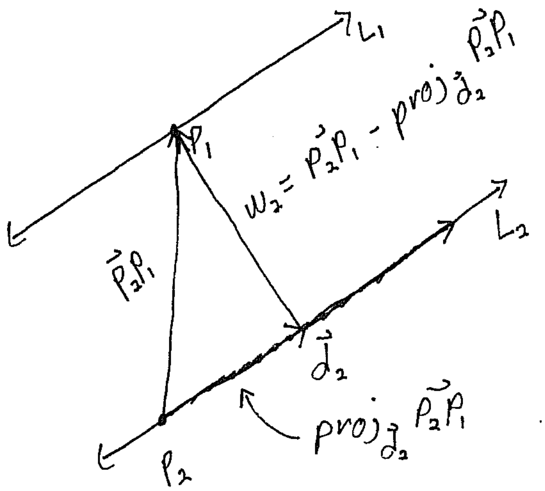
$$\text{proj}_{\vec{n}} \vec{P_0P} = \frac{\vec{P_0P}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$$

$$= \frac{(4,0,0)\cdot(-2,1,-1)}{(-2,1,-1)\cdot(-2,1,-1)}(-2,1,-1)$$

$$= \frac{-8}{6}(-2,1,-1) = \left(\frac{16}{6}, \frac{-8}{6}, \frac{8}{6}\right)$$

$$\therefore \text{distance} = \|\text{proj}_{\vec{n}} \vec{P_0P}\| = \left\|\left(\frac{16}{6}, \frac{-8}{6}, \frac{8}{6}\right)\right\| = \sqrt{\left(\frac{16}{6}\right)^2+\left(\frac{-8}{6}\right)^2+\left(\frac{8}{6}\right)^2} = \sqrt{\frac{384}{36}} = \sqrt{\frac{32}{3}}$$

Question 5. *(5 marks)* Find the distance between the two parallel lines $\underbrace{(x,y,z) = \underbrace{(2,2,0)}_{P_1}+t\underbrace{(0,2,4)}_{\vec{d}_1}}_{L_1}$ and $\underbrace{(x,y,z) = \underbrace{(0,1,2)}_{P_2}+s\underbrace{(0,1,2)}_{\vec{d}_2}}_{L_2}$ *(using projections)*



$\vec{P_2P_1} = P_1 - P_2 = (2,2,0)-(0,1,2)=(2,1,-2)$

$$\text{proj}_{\vec{d}_2} \vec{P_2P_1} = \frac{\vec{P_2P_1}\cdot\vec{d}_2}{\vec{d}_2\cdot\vec{d}_2}\vec{d}_2$$

$$= \frac{(2,1,-2)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}(0,1,2)$$

$$= \frac{-3}{1+4}(0,1,2) = \left(0, \frac{-3}{5}, \frac{-6}{5}\right)$$

$$W_2 = \vec{P_2P_1} - \text{proj}_{\vec{d}_2} \vec{P_2P_1}$$

$$= (2,1,-2) - \left(0,\frac{-3}{5},\frac{-6}{5}\right) = \left(2, \frac{8}{5}, \frac{-4}{5}\right)$$

$$\therefore \text{distance} = \|\vec{W}_2\|$$

$$= \left\|\left(2,\frac{8}{5},\frac{-4}{5}\right)\right\| = \sqrt{\left(\frac{10}{5}\right)^2+\left(\frac{8}{5}\right)^2+\left(\frac{-4}{5}\right)^2} = \sqrt{\frac{180}{25}} = \sqrt{\frac{36}{5}}$$

Question 6. Given the following two planes $x-y+3z=4$ and $-2x+y+3z=5$.

a. *(4 marks)* Find the parametric equation of the line of intersection of the two planes.

b. *(2 marks)* Find the equation of the line passing through the point $(2,0,1)$ and parallel to the intersection of the two planes.

a)
$$\begin{bmatrix} 1 & -1 & 3 & 4 \\ -2 & 1 & 3 & 5 \end{bmatrix} \underset{2R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & -1 & 9 & 13 \end{bmatrix}$$

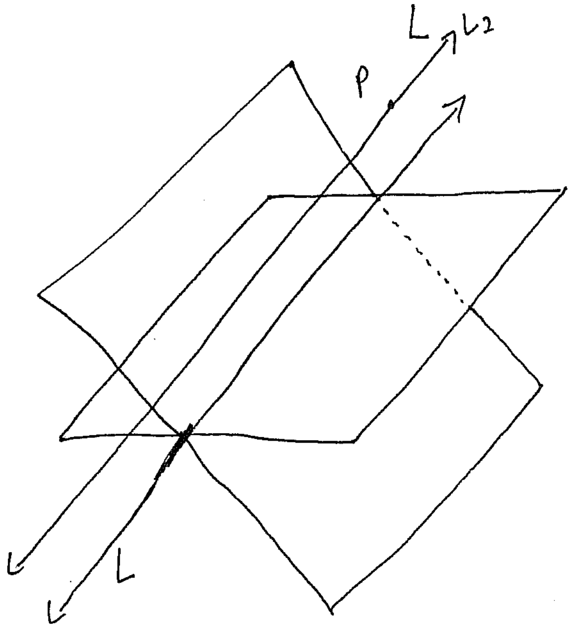
$$\sim \begin{array}{r} -R_2+R_1 \to R_1 \\ -R_2 \end{array} \begin{bmatrix} 1 & 0 & -6 & -9 \\ 0 & 1 & -9 & -13 \end{bmatrix}$$

let $z=t$

$$\begin{aligned} x-6t&=-9 \\ y-9t&=13 \\ z&=t \end{aligned} \iff \begin{aligned} x&=-9+6t \\ y&=-13+9t \\ z&=t \end{aligned}$$

$$\therefore \underbrace{(x,y,z)=(-9,-13,0)+t(6,9,1)}$$

b)



L_2:

$(x,y,z)=(2,0,1)+t(6,9,1)$

Question 7. *(6 marks)* Minimize $C = x + y + 2z$ subject to

$$P = -C$$
$$P = -(x + y + 2z)$$
$$P = -x - y - 2z$$

$$x + y + z \geq 10$$
$$2x + 4y + 2z \geq 30$$

$$\Updownarrow$$

$$x + y + 2z + P = 0$$

$$\begin{aligned} x + y + z - S_1 \quad\quad &= 10 \\ 2x + 4y + 2z \quad - S_2 \quad &= 30 \\ x + y + 2z \quad\quad + P &= 0 \end{aligned}$$

pivot column

$$\left[\begin{array}{ccccccc} 1 & 1 & 1 & -1 & 0 & 0 & 10 \\ 2 & 4 & 2 & 0 & -1 & 0 & 30 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} r = 10/1 = 10 \leftarrow \text{pivot row} \\ r = 30/2 = 15 \\ \end{array}$$

pivot column

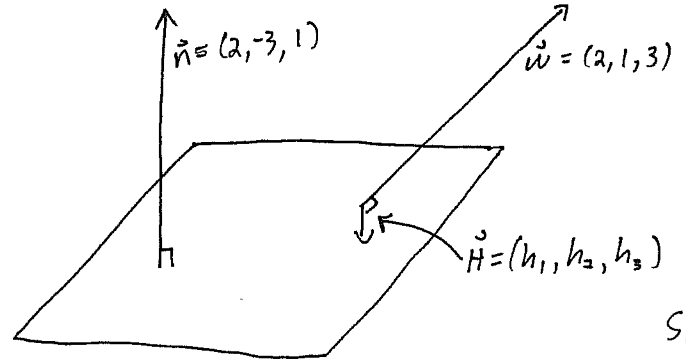
$$\begin{array}{l} \\ -2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccccccc} 1 & 1 & 1 & -1 & 0 & 0 & 10 \\ 0 & 2 & 0 & 2 & -1 & 0 & 10 \\ 0 & 0 & 1 & 1 & 0 & 1 & -10 \end{array}\right] \begin{array}{l} r = 10/1 = 10 \\ r = \frac{10}{2} = 5 \leftarrow \text{pivot row} \\ \end{array}$$

$$\frac{1}{2} R_2 \left[\begin{array}{ccccccc} 1 & 1 & 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 0 & 5 \\ 0 & 0 & 1 & 1 & 0 & 1 & -10 \end{array}\right]$$

$$-R_2 + R_1 \to R_1 \left[\begin{array}{ccccccc} 1 & 0 & 1 & -2 & \frac{1}{2} & 0 & 5 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 0 & 5 \\ 0 & 0 & 1 & 1 & 0 & 1 & -10 \end{array}\right]$$

$$\therefore \min \; C = -P = -(-10) = 10 \quad \text{at} \quad \begin{array}{l} x = 5 \\ y = 5 \\ z = 0 \end{array}$$

Bonus Question. *(3 marks)* Find all unit vectors lying on the plane $2x-3y+z=4$ and orthogonal to the vector $\mathbf{w}=(2,1,3)$.



$$\vec{n}\cdot\vec{H}=0$$
$$\vec{w}\cdot\vec{H}=0$$

So,

$$(2,-3,1)\cdot(h_1,h_2,h_3)=0$$
$$(2,1,3)\cdot(h_1,h_2,h_3)=0$$

$$\Leftrightarrow \quad \begin{aligned} 2h_1-3h_2+h_3&=0\\ 2h_1+h_2+3h_3&=0\end{aligned}$$

$$\begin{bmatrix} 2 & -3 & 1 & 0\\ 2 & 1 & 3 & 0\end{bmatrix} \sim_{-R_1+R_2\to R_2} \begin{bmatrix} 2 & -3 & 1 & 0\\ 0 & 4 & 2 & 0\end{bmatrix}$$

$$\sim_{4R_1} \begin{bmatrix} 8 & -12 & 4 & 0\\ 0 & 4 & 2 & 0\end{bmatrix} \sim_{3R_2+R_1\to R_1} \begin{bmatrix} 8 & 0 & 10 & 0\\ 0 & 4 & 2 & 0\end{bmatrix}$$

$$\sim_{\substack{\frac{1}{8}R_1\\ \frac{1}{4}R_2}} \begin{bmatrix} 1 & 0 & \frac{10}{8} & 0\\ 0 & 1 & \frac{1}{2} & 0\end{bmatrix}$$

$$z=t$$
$$x+\frac{5}{4}t=0$$
$$y+\frac{1}{2}t=0$$

$$\therefore \quad x=-\frac{5}{4}t,\quad y=-\frac{1}{2}t,\quad z=t$$

$$\therefore \vec{H}(h_1,h_2,h_3)=\left(-\frac{5}{4}t,-\frac{1}{2}t,t\right)$$
$$=t\left(-\frac{5}{4},-\frac{1}{2},1\right)$$
$$=t(-5,-2,4)$$

$\therefore$ unit vector

$$=\frac{\vec{H}}{\|\vec{H}\|}$$

$$=\frac{t(-5,-2,4)}{\sqrt{(-5t)^2+(-2t)^2+(4t)^2}}$$

$$=\frac{t(-5,-2,4)}{\sqrt{t^2(25+4+16)}}$$

$$=\frac{\pm\cancel{t}(-5,-2,4)}{\cancel{t}\sqrt{45}}=\pm\left(\frac{-5}{\sqrt{45}},\frac{-2}{\sqrt{45}},\frac{4}{\sqrt{45}}\right)$$

Note: The alternate solution is much better.

Alternate Solution:

We want a vector parallel to the plane so a vector orthogonal to the normal. And we want a vector orthogonal to the vector $\vec{w} = (2,1,3)$. Hence a vector $\vec{H}$ orthogonal to both $\vec{w} = (2,1,3)$ and $\vec{n} = (2,-3,1)$

$$\therefore \quad \vec{H} = \underset{\begin{smallmatrix} 2 & 2 \\ 1 & -3 \\ 3 & 1 \end{smallmatrix}}{\vec{w} \times \vec{n}} = \left(\begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 1 & -3 \end{vmatrix} \right) = (10, 4, -8)$$

But we wanted a unit vector so

$$\vec{u} = \frac{\vec{H}}{\|\vec{H}\|} = \frac{(10,4,-8)}{\sqrt{10^2+4^2+(-8)^2}}$$

$$= \frac{(10,4,-8)}{\sqrt{180}}$$

$$= \left(\frac{5}{\sqrt{45}}, \frac{2}{\sqrt{45}}, \frac{-4}{\sqrt{45}} \right)$$

and the unit vector in the opposite direction is also a good solution $\therefore \quad \vec{u} = \pm \left(\frac{5}{\sqrt{45}}, \frac{2}{\sqrt{45}}, \frac{-4}{\sqrt{45}} \right)$