

Test 3

This test is graded out of 30 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (3 marks) Find the volume of the parallelepiped defined by $\mathbf{u} = (2, 0, -3)$, $\mathbf{v} = (-1, 1, -1)$ and $\mathbf{w} = (1, 0, 3)$.

$$\begin{aligned}\vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{vmatrix} 2 & 0 & -3 \\ -1 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix} = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} \\ &= (-1) \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix} \\ &= 6 + 3 = 9\end{aligned}$$

∴ the volume is 9.

Question 2. (3 marks) Find a vector perpendicular to both $\mathbf{u} = (2, 1, -3)$ and $\mathbf{v} = (-1, 1, -1)$.

$$\begin{aligned}\vec{u} \times \vec{v} &= \left(\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix}, - \begin{vmatrix} 2 & -1 \\ -3 & -1 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \right) \\ &\stackrel{2 \quad -1}{=} \begin{pmatrix} 2, & 5, & 3 \end{pmatrix}\end{aligned}$$

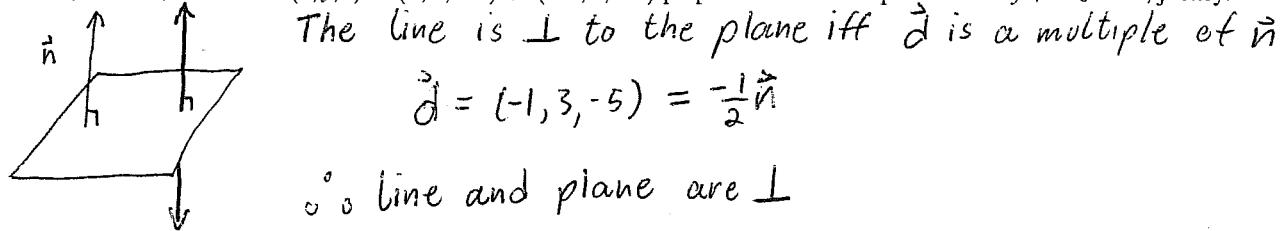
Question 3. (3 marks) Determine if the line $(x, y, z) = (1, 0, 2) + t(1, 2, 2)$ passes through the point $(2, 2, 3)$.

$$\text{So } L \quad \begin{cases} x = 1+t \\ y = 2t \\ z = 2+2t \end{cases} \quad \text{and} \quad \begin{aligned} 2 &= 1+t \Rightarrow t = 1 \\ 2 &= 2t \Rightarrow t = 1 \\ 3 &= 2+2t \Rightarrow t = \frac{1}{2} \end{aligned}$$



∴ the point is not on the line.

Question 4. (2 marks) Is the line $(x, y, z) = (1, 2, -2) + t(-1, 3, -5)$ perpendicular to the plane $2x - 6y + 10z = 11$, justify.



Question 5. (2 marks) Is the plane $-x + y - 2z = 31$ perpendicular to the plane $4x + 2y - z = 10$, justify.

The planes are \perp iff their normals are \perp

$$\vec{n}_1 = (-1, 1, -2)$$

$$\vec{n}_2 = (4, 2, -1)$$

$$\vec{n}_1 \cdot \vec{n}_2 = (-1)(4) + (1)(2) + (-2)(-1) = 0$$

\therefore the planes are \perp

Question 6. (3 marks) Find the equation of the plane passing through the points $A(1, 2, -3)$, $B(2, 1, 0)$ and $C(0, 1, -3)$.

$$\vec{AB} = B - A = (2, 1, 0) - (1, 2, -3) = (1, -1, 3)$$

$$\vec{AC} = C - A = (0, 1, -3) - (1, 2, -3) = (-1, -1, 0)$$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} 1 & -1 & 3 \\ -1 & -1 & 0 \\ 0 & 1 & -3 \end{vmatrix} = (3, -3, -2)$$

$$\therefore 3x - 3y - 2z = d$$

$$3(0) - 3(1) - 2(-3) = d$$

$$3 = d$$

$\therefore 3x - 3y - 2z = 3$

Question 7. (4 marks) Determine the distance between the point $P(2, 1, 1)$ and the plane $-2x + y - z = 10$. (using projections)

distance = $\|\text{proj}_{\vec{n}} \vec{P_0 P}\|$

Lets find P_0 on the plane, let $y=0$ $z=0$

$$\therefore -2x + 0 - 0 = 10 \quad x = -5 \quad \therefore P_0(-5, 0, 0)$$

$$\vec{P_0 P} = P - P_0 = (2, 1, 1) - (-5, 0, 0) = (7, 1, 1)$$

$$\text{proj}_{\vec{n}} \vec{P_0 P} = \frac{\vec{P_0 P} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n}$$

$$= \frac{(7, 1, 1) \cdot (-2, 1, -1)}{(-2, 1, -1) \cdot (-2, 1, -1)} (-2, 1, -1)$$

$$= \frac{-14}{6} (-2, 1, -1)$$

$$= \left(\frac{14}{3}, \frac{-7}{3}, \frac{7}{3} \right)$$

$$\text{distance} = \left\| \left(\frac{14}{3}, \frac{-7}{3}, \frac{7}{3} \right) \right\| = \sqrt{\left(\frac{14}{3} \right)^2 + \left(\frac{-7}{3} \right)^2 + \left(\frac{7}{3} \right)^2} = \sqrt{\frac{98}{3}}$$

Question 8. Given the following two planes $x - y + 3z = 2$ and $-2x + y + 3z = 3$.

- (4 marks) Find the parametric equation of the line of intersection of the two planes.
- (3 marks) Find the equation of the plane that contains the point $(2, 0, 1)$ and the intersection of the plane.

a)

$$\begin{bmatrix} 1 & -1 & 3 & 2 \\ -2 & 1 & 3 & 3 \end{bmatrix} \sim \begin{array}{l} 2R_1 + R_2 \rightarrow R_2 \\ -R_2 \end{array} \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & -1 & 9 & 7 \end{bmatrix}$$

$$\sim \begin{array}{l} -R_2 + R_1 \rightarrow R_1 \\ -R_2 \end{array} \begin{bmatrix} 1 & 0 & -6 & -5 \\ 0 & 1 & -9 & -7 \end{bmatrix}$$

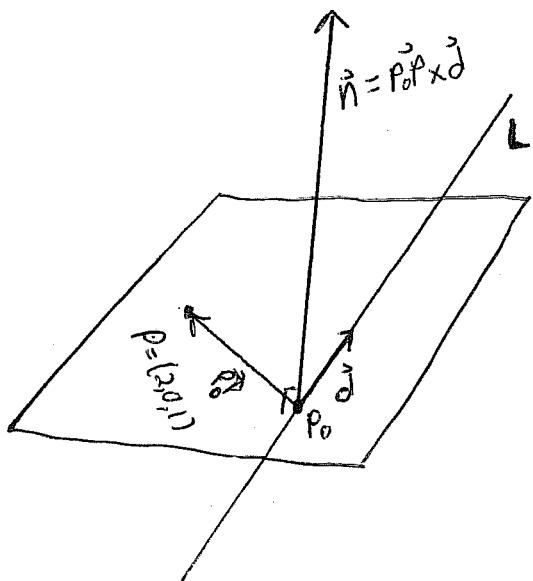
let $z = t$

$$\begin{aligned} x - 6t &= -5 & x &= -5 + 6t \\ y - 9t &= -7 \Leftrightarrow & y &= -7 + 9t \\ z &= t & z &= t \end{aligned}$$

$$\therefore (x, y, z) = (-5, -7, 0) + t(6, 9, 1)$$

\curvearrowleft point on the line \curvearrowleft direction vector

b)



$$\vec{P_0 P} = \vec{P} - \vec{P_0} = (2, 0, 1) - (-5, 7, 0) = (7, -7, 1)$$

$$\begin{aligned} \vec{n} &= \vec{P_0 P} \times \vec{d} = \begin{vmatrix} 1 & 7 & -9 \\ 7 & 1 & 6 \\ 1 & -1 & 1 \end{vmatrix} = (1, 1, 1) \\ &= (2, 1, -21) \end{aligned}$$

$$\begin{aligned} \therefore 2x + y - 21z &= d \\ 2(2) + 0 - 21(1) &= d \\ -17 &= d \end{aligned}$$

$$\therefore 2x + y - 21z = -17$$

Question 9. (5 marks) Maximize $P = 4x + 5y + 4z$ subject to

$$\begin{array}{l} -x + y + 2z \leq 40 \\ 2x - y - z \leq 10 \\ \hline \end{array} \Leftrightarrow \begin{array}{rcl} -x + y + 2z + S_1 & & = 40 \\ 2x - y - z + S_2 & & = 10 \\ -4x - 5y - 4z & & P = 0 \end{array}$$

\swarrow pivot

$$\left[\begin{array}{cccccc} -1 & 1 & 2 & 1 & 0 & 0 & 40 \\ 2 & -1 & -1 & 0 & 1 & 0 & 10 \\ -4 & -5 & -4 & 0 & 0 & 1 & 0 \end{array} \right] \leftarrow \text{pivot row}$$

\uparrow pivot column

$$\begin{array}{l} R_1 + R_2 \rightarrow R_2 \\ 5R_1 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{cccccc} -1 & 1 & 2 & 1 & 0 & 0 & 40 \\ 1 & 0 & 1 & 1 & 1 & 0 & 50 \\ -9 & 0 & 6 & 5 & 0 & 1 & 200 \end{array} \right] \leftarrow \text{pivot row}$$

\uparrow pivot column

$$\begin{array}{l} R_2 + R_1 \rightarrow R_1 \\ 9R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{cccccc} 0 & 1 & 3 & 2 & 1 & 0 & 90 \\ 1 & 0 & 1 & 1 & 1 & 0 & 50 \\ 0 & 0 & 15 & 14 & 9 & 1 & 650 \end{array} \right]$$

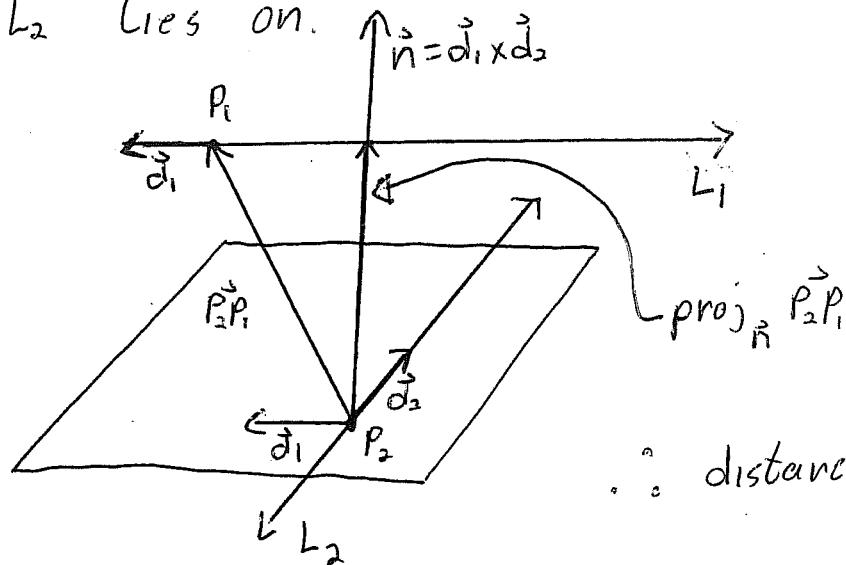
\therefore max is 650 at $x = 50$
 $y = 90$
 $z = 0$

Bonus Question. (3 marks) Find the distance between the two lines $(x, y, z) = (2+t, -3t, 1)$ and $(x, y, z) = (t, 1-t, 3+2t)$.

$$\text{Let } L_1: (x, y, z) = (2, 0, 1) + t(1, -3, 0)$$

$$L_2: (x, y, z) = (0, 1, 3) + t(1, -1, 2)$$

Create a plane that is parallel to L_1 and which L_2 lies on.



$$\therefore \text{distance} = \|\text{proj}_{\vec{n}} \vec{P_2P_1}\|$$

$$\vec{P_2P_1} = P_1 - P_2 = (0, 1, 3) - (2, 0, 1) = (-2, 1, 2)$$

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = (-6, -2, 2)$$

$$\text{proj}_{\vec{n}} \vec{P_2P_1} = \frac{\vec{P_2P_1} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n}$$

$$= \frac{(-2, 1, 2) \cdot (-6, -2, 2)}{(-6, -2, 2) \cdot (-6, -2, 2)} (-6, -2, 2)$$

$$= \frac{12 - 2 + 4}{44} (-6, -2, 2) = \left(-\frac{21}{11}, -\frac{7}{11}, \frac{7}{11} \right)$$

$$\therefore \text{distance} = \left\| \left(-\frac{21}{11}, -\frac{7}{11}, \frac{7}{11} \right) \right\| = \sqrt{\left(\frac{-21}{11} \right)^2 + \left(\frac{-7}{11} \right)^2 + \left(\frac{7}{11} \right)^2}$$

$$= \sqrt{\frac{49}{11}} = \frac{7}{\sqrt{11}}$$