Dawson College: Calculus I: 201-NYA-05 13

November 26, 2010

Last Name: SOLUTIONS

First Name: ______

Student ID: ______

Test 3A

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to use **correct notation**.

Question 1. *(5 marks)*
(a) Find $\frac{dy}{dx}$ for $(x-y^2)^{10}+y-x^3=0$.
(b) Find the tangent line to this curve at the point (1,0).

a) $\frac{d}{dx}(x-y^2)^{10} + \frac{d}{dx}(y) - \frac{d}{dx}(x^3) = 0$

$10(x-y^2)^9(1-2yy') + y' - 3x^2 = 0$

$10(x-y^2)^9 - 10(x^2-y^2)^9(2yy') + y' - 3x^2 = 0$

$y' - 20yy'(x^2-y^2)^9 = 3x^2 - 10(x-y^2)^9$

$y'[1 - 20y(x^2-y^2)^9] = 3x^2 - 10(x-y^2)^9$

$$y' = \frac{3x^2 - 10(x-y^2)^9}{1 - 20y(x^2-y^2)^9}$$

b) $$m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = \frac{3(1) - 10(1-0^2)^9}{1 - 20(0)(1-0)^9} = \frac{-7}{1} = -7$$

$y = mx + b$

$0 = -7(1) + b$

$7 = b$

$\therefore y = -7x + 7$

Question 2. *(8 marks)* Suppose the demand equation for a certain product is

$$200x^2 + 11P^2 = 4200$$

where x represents the number of units demanded each week when the unit price is $\$p$. How much is the quantity demanded increasing when the unit price is \$13 per unit and unit price is decreasing at a rate of \$0.22 per unit per week? (You can use decimals, round your final answer to the nearest unit.)

$x = ?$ $\frac{dx}{dt} = ?$ $p = 13$, $\frac{dp}{dt} = -0.22$

$$200x^2 + 11(13)^2 = 4200$$

$$200x^2 = 4200 - 11(13)^2$$

$$x^2 = \frac{4200 - 11(13)^2}{200}$$

$$x = \sqrt{\frac{4200 - 11(13)^2}{200}} = 3.4213$$

$$\frac{d}{dt}(200x^2) + \frac{d}{dt}(11p^2) = \frac{d}{dt}(4200)$$

$$400x\frac{dx}{dt} + 22p\frac{dp}{dt} = 0$$

$$400(3.4213)\frac{dx}{dt} + 22(13)(-0.22) = 0$$

$$1368.503\frac{dx}{dt} = 62.92$$

$$\frac{dx}{dt} = 0.04598$$

$\therefore$ QUANTITY DEMANDED IS INCREASING BY 46 UNITS PER WEEK AT THE TIME UNDER CONSIDERATION.

Question 3. *(6 marks)* Find the asymptotes of the following functions (you do not need to sketch the graphs).

(a) $f(x) = \dfrac{3x^2 - 3x + 1}{x^2 + 4x + 3}$

$$\lim_{x \to \pm\infty} \frac{3x^2 - 3x + 1}{x^2 + 4x + 3} = \lim_{x \to \pm\infty} \frac{3 - 3/x + 1/x^2}{1 + 4/x + 3/x^2} = \frac{3}{1} = 3$$

$\therefore$ $y = 3$ IS A HORIZONTAL ASYMPTOTE.

$x^2 + 4x + 3 = 0 \Rightarrow (x+1)(x+3) = 0 \Rightarrow x = -1, -3$

$3(-1)^2 - 3(-1) + 1 \neq 0$

$3(-3)^2 - 3(-3) + 1 \neq 0$

$\therefore$ $x = -1$ AND $x = -3$ ARE VERTICAL ASYMPTOTES.

(b) $g(x) = \dfrac{5}{x^3 - x}$

$$\lim_{x \to \pm\infty} \frac{5}{x^3 - x} = \lim_{x \to \pm\infty} \frac{5/x^3}{1 - 1/x^2} = 0$$

$\therefore$ $y = 0$ IS A HORIZONTAL ASYMPTOTE.

$x^3 - x = 0$

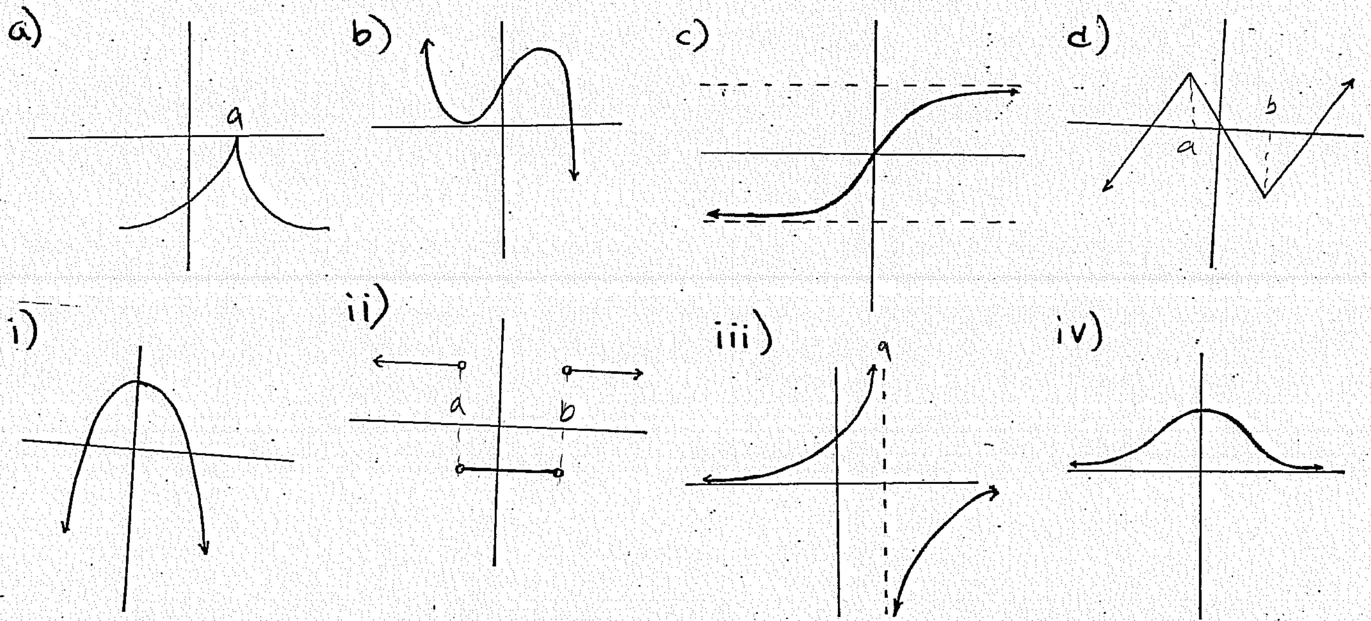
$x(x^2 - 1) = 0$

$x(x+1)(x-1) = 0$

$\therefore$ $x = 0, -1, 1$

$\therefore$ $x = -1$, $x = 0$, $x = 1$ ARE VERTICAL ASYMPTOTES.

Question 4. *(6 marks)* Match the graph of each function from a, b, c, d with the graph of its derivative from i, ii, iii, iv. Give a brief mathematical explanation for each match.



iii) IS THE DERIVATIVE OF a) BECAUSE $f'(x)$ DOES NOT EXIST AT a AND $f(x)$ HAS A CORNER AT a

i) IS THE DERIVATIVE OF b) BECAUSE $f'(x) > 0$ WHERE f IS INCREASING AND $f'(x) < 0$ WHERE f IS DECREASING.

iv) IS THE DERIVATIVE OF c) BECAUSE f IS ALWAYS INCREASING AND $f'(x) > 0$ EVERYWHERE.

ii) IS THE DERIVATIVE OF d) BECAUSE f HAS CORNERS AT a AND b WHERE $f'(x)$ DOES NOT EXIST

Question 5. *(10 marks)* Do the first six steps of the seven step procedure for graphing the curve

$$f(x) = x^6 - 6x^5$$

You do not need to do the seventh step, that is, **you do not need to graph this curve.**

1) DOMAIN: $\mathbb{R}$ 2) y-int: $x=0 \Rightarrow y = 0^6 + 5(0)^2 = 0 \quad \therefore (0,0)$

x-int: $y=0 \Rightarrow 0 = x^6 - 6x^5 = x^5(x-6)$

$\Rightarrow x = 0, 6$

$\therefore (0,0), (6,0)$

3) $\lim\limits_{x \to \pm\infty} x^6 + 6x^5 = \infty$ NO H.A.

4) NO V.A. SINCE f IS A POLYNOMIAL.

5) $f'(x) = 6x^5 - 30x^4 = 0$

$6x^4(x-5) = 0$

$\therefore x = 0, 5$

TEST POINTS:

$x = -1 \quad f'(-1) = -36 < 0$

$x = 1 \quad f(1) = -24 < 0$

$x = 6 \quad f(6) = 7776 > 0$

$f(x)$	$\searrow$		$\searrow$		$\nearrow$
$f'(x)$	$-$		$-$		$+$
x		0		5	

f IS DECREASING ON $(-\infty, 0)$ AND $(0, 5)$

f IS INCREASING ON $(5, \infty)$

$f(5) = -3125$ IS A RELATIVE MINIMUM

6) $f''(x) = 30x^4 - 120x^3 = 0$

$30x^3(x-4) = 0$

$\therefore x = 0, 4$

TEST POINTS:

$x = -1 \quad f''(-1) = 150 > 0$

$x = 1 \quad f''(1) = -90 < 0$

$x = 5 \quad f''(5) = 3750 > 0$

$f(x)$	$\cup$		$\cap$		$\cup$
$f''(x)$	$+$		$-$		$+$
x		0		4	

f IS CONCAVE UPWARD ON $(-\infty, -4)$ AND $(0, \infty)$.

f IS CONCAVE DOWNWARD ON $(0, 4)$

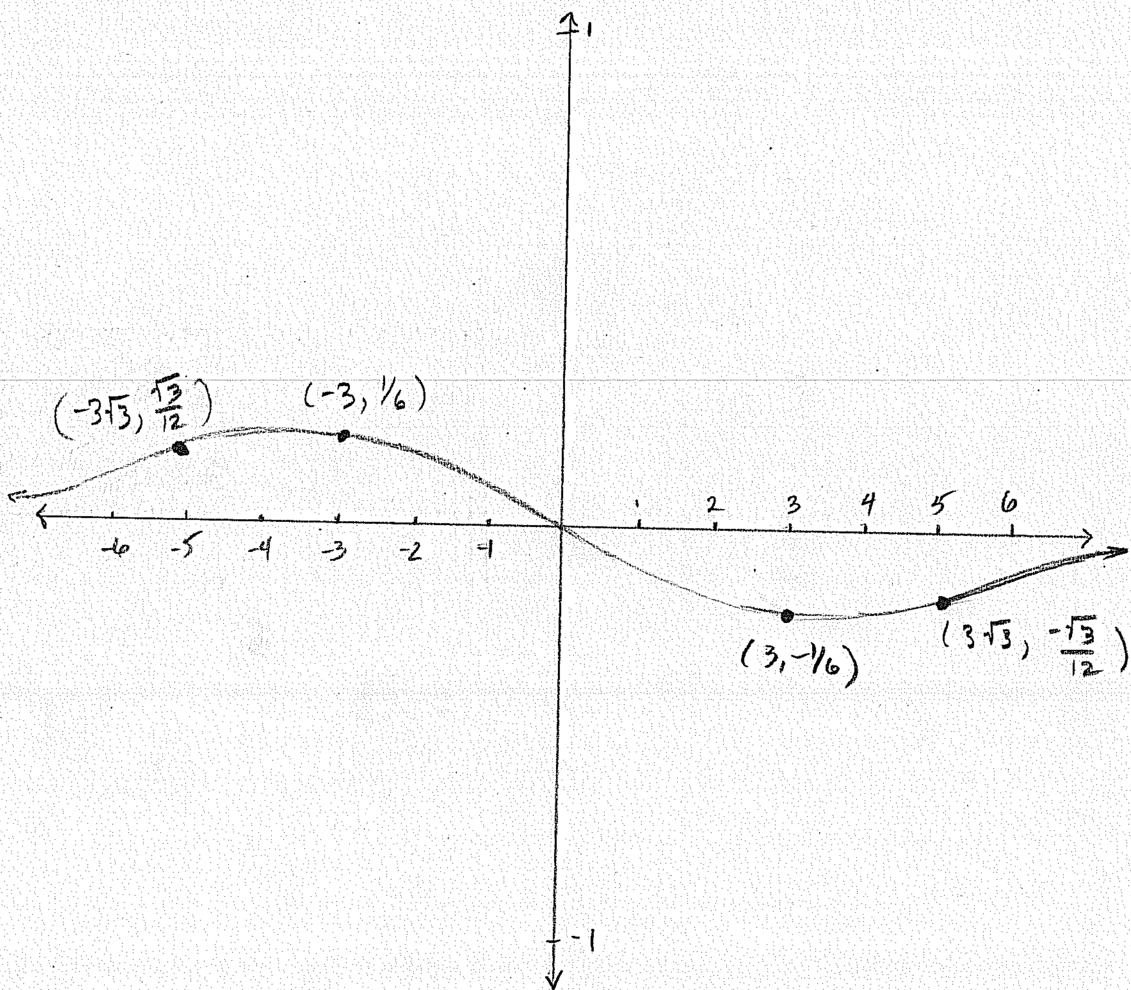
$f(0) = 0$, $f'(0) = 0 \Rightarrow (0,0)$ IS AN INFLECTION POINT

$f(4) = -2048$, $f'(4) = -1536 \Rightarrow (4, -2048)$ IS AN INFLECTION POINT.

Question 6. *(5 marks.)* The function

$$f(x) = -\frac{x}{x^2+9}$$

has the following properties:
Domain: $(-\infty,\infty)$. **x-intercept, y-intercept**: $(0,0)$.
Horizontal asymptote: $y=0$. **Vertical asymptote:** None.
f **is decreasing on:** $(-3,3)$. f **is increasing on:** $(-\infty,-3)$ and $(+3,\infty)$.
Relative minimum: $(3,-1/6)$. **Relative maximum:** $(-3,1/6)$
Concave downward on: $(-3\sqrt{3},0)$ and $(3\sqrt{3},\infty)$.
Concave upward on: $(-\infty,-3\sqrt{3})$ and $(0,3\sqrt{3})$.
Inflection Points: $(3\sqrt{3},-\sqrt{3}/12)$ and $(-3\sqrt{3},+\sqrt{3}/12)$ Sketch the graph of $f(x)$.



Question 7. *(5 marks.)* Use the second derivative test to find the local extrema for:

$$f(x) = x + \frac{9}{x}$$

$f'(x) = 1 - \frac{9}{x^2}$

$= \frac{x^2-9}{x^2}$

$f'(x) = 0$

$\Rightarrow x^2 - 9 = 0$

$(x+3)(x-3) = 0$

$x = \pm 3$

$f'(x)$ D.N.E

$\Rightarrow x = 0$

BUT $f(0)$ IS NOT DEFINED SO NO EXTREMA AT $x=0$

$f''(x) = \frac{18}{x^3}$

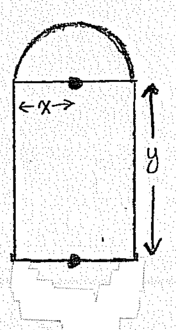
$f''(3) = \frac{18}{(3)^3} = \frac{1}{3} > 0$ $\therefore$ f IS CONCAVE UPWARD AT $x=3$

$\therefore$ $f(3) = 3 + \frac{9}{3} = 6$ IS A RELATIVE MAXIMUM.

$f''(-3) = \frac{18}{(-3)^3} = -\frac{1}{3} < 0$ $\therefore$ f IS CONCAVE DOWNWARD AT $x = -3$

$\therefore$ $f(-3) = -3 + \frac{9}{-3} = -6$ IS A RELATIVE MINIMUM.

Question 8. *(8 marks)* A window has the shape of one semicircle above and one semicircle below a rectangle. If the window is to have a outer perimeter of 24ft, what should its dimensions be to allow the maximum amount of light through the window?

1. 

2. $A = \frac{1}{2}\pi x^2 + 2xy$

3. $\frac{1}{2}(2\pi x) + 2y + x = P = 24$

$\pi x + 2y + 2x = 24$

$2y = 24 - 2x - \pi x$

$y = 12 - x - \frac{\pi}{2}x$

$\therefore \quad A = \frac{1}{2}\pi x^2 + 2x(12 - x - \frac{\pi}{2}x)$

$= \frac{\pi}{2}x^2 + 24x - 2x^2 - \pi x^2 = 24x - (2 + \frac{\pi}{2})x^2 = f(x)$

RESTRICTIONS:

$x \geq 0$

$12 - x - \frac{\pi}{2}x \geq 0 \quad (y > 0)$

$12 \geq x + \frac{\pi}{2}x$

$24 \geq 2x + \pi x$

$24 \geq (2+\pi)x$

$\frac{24}{2+\pi} \geq x$

$\therefore$ THE DOMAIN OF f IS $[0, \frac{24}{2+\pi}]$

4. $f'(x) = 24 - 2(2 + \frac{\pi}{2})x = 0$

$24 = (4+\pi)x$

$\frac{24}{4+\pi} = x$

END POINTS:

$f(0) = 0$

$f(\frac{24}{2+\pi}) = 68.654$

C.N.

$f(\frac{24}{4+\pi}) = 95.360 \leftarrow$ MAXIMUM.

$\therefore \quad x = \frac{24}{4+\pi} \approx 3.36$

$y = 12 - \left(\frac{24}{4+\pi}\right) - \frac{\pi}{2}\left(\frac{24}{4+\pi}\right) \approx 3.36$