Dawson College: Calculus I: 201-NYA-05 13 November 26, 2010

Last Name: SOLUTIONS.

First Name:

Student ID:

Test 3B

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to use **correct notation**.

Question 1. *(5 marks)*
(a) Find $\frac{dy}{dx}$ for $(x-y^2)^{10}+y=x^3$.
(b) Find the tangent line to this curve at the point $(1,0)$.

a) $\frac{d}{dx}(x-y^2)^{10} + \frac{d}{dx}(y) = \frac{d}{dx}(x^3)$

$10(x-y^2)^9 \cdot \frac{d}{dx}(x-y^2) + y' = 3x^2$

$10(x-y^2)^9(1-2yy') + y' = 3x^2$

$10(x-y^2)^9 - 20yy'(x-y^2)^9 + y' = 3x^2$

$-20yy'(x-y^2)^9 + y' = 3x^2 - 10(x-y^2)^9$

$y'[-20y(x-y^2)^9 + 1] = 3x^2 - 10(x-y^2)^9$

$\frac{dy}{dx} = y' = \dfrac{3x^2-10(x-y^2)^9}{1-20y(x-y^2)}$

b) $m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = \dfrac{3(1)^2-10(1-0^2)^9}{1-20(0)(1-0^2)^9} = \dfrac{3-10}{1} = -7$

$\therefore$ $y = mx+b$

$0 = -7(1)+b$

$7 = b$

$\therefore$ $\boxed{y = -7x+7}$

Question 2. *(8 marks)* Suppose the demand equation for a certain product is

$$220x^2 + 12P^2 = 4300$$

where x represents the number of units demanded each week when the unit price is $\$p$. How much is the quantity demanded increasing when the unit price is \$12 per unit and unit price is decreasing at a rate of \$0.20 per unit per week? (You can use decimals, round your final answer to the nearest unit.) X IS IN UNITS OF ONE THOUSAND

$$220x^2 + 12(12)^2 = 4300$$

$$220x^2 = 4300 - 12^3$$

$$x^2 = \frac{4300 - 12^3}{220}$$

$$x = \sqrt{\frac{4300 - 12^3}{220}} = 3.41920$$

$$\frac{d}{dt}(220x^2) + \frac{d}{dt}(12p^2) = \frac{d}{dt}(4300)$$

$$440x\frac{dx}{dt} + 24p\frac{dp}{dt} = 0$$

$$440(3.41920)\frac{dx}{dt} + 24(12)(-0.20) = 0$$

$$1504.447\frac{dx}{dt} = 57.6$$

$$\frac{dx}{dt} = 0.0383$$

∴ QUANTITY DEMANDED IS INCREASING BY 38 units PER WEEK AT THE TIME UNDER CONSIDERATION.

Question 3. *(6 marks)* Find the asymptotes of the following functions (you do not need to sketch the graphs).

(a) $f(x) = \dfrac{7}{x^3 - x}$

$\lim_{x \to \pm\infty} \dfrac{7}{x^3 - x} = \lim_{x \to \pm\infty} \dfrac{7/x^3}{1 - 1/x^2} = 0$

$\therefore y = 0$ IS A HORIZONTAL ASYMPTOTE

$x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x(x+1)(x-1) = 0$

$x = 0$, $x = -1$ AND $x = 1$ ARE VERTICAL ASYMPTOTES.

(b) $g(x) = \dfrac{4x^2 + 2x - 2}{x^2 + 6x + 5}$

$\lim_{x \to \pm\infty} \dfrac{4x^2 + 2x - 2}{x^2 + 6x + 5} = \lim_{x \to \pm\infty} \dfrac{4 + 2/x - 2/x^2}{1 + 6/x + 5/x^2} = 4$

$\therefore y = 4$ IS A HORIZONTAL ASYMPTOTE.

NOW, $x^2 + 6x + 5 = 0 \Rightarrow (x+5)(x+1) = 0 \Rightarrow x = -5, -1$

$4(-5)^2 + 2(-5) - 2 \neq 0$

$\therefore x = -5$ IS A VERTICAL ASYMPTOTE

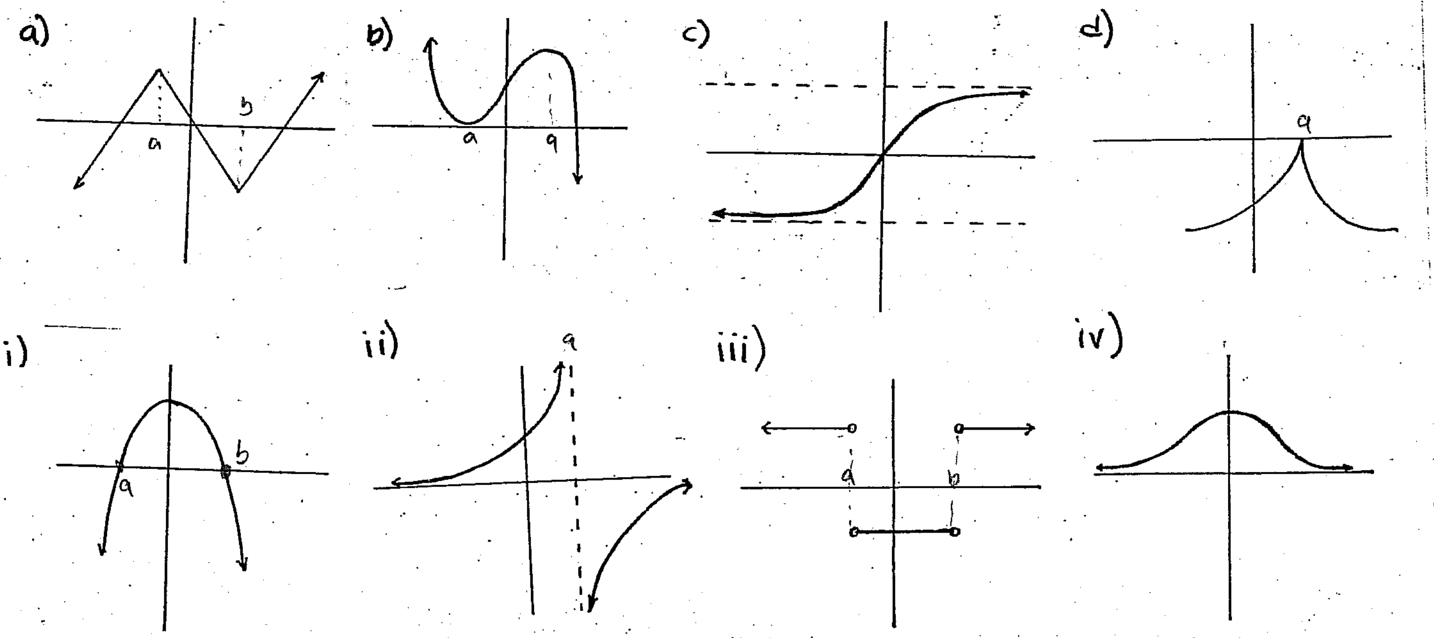
AND

$4(-1)^2 + 2(-1) - 2 = 0$

$\lim_{x \to -1} \dfrac{4x^2 + 2x - 2}{x^2 + 6x + 5} = \lim_{x \to -1} \dfrac{(4x-2)(x+1)}{(x+5)(x+1)} = \lim_{x \to -1} \dfrac{4x-2}{x+5} = -\dfrac{10}{4}$

$\therefore x = -1$ IS NOT A V.A.

Question 4. *(6 marks)* Match the graph of each function from a, b, c, d with the graph of its derivative from i, ii, iii, iv. Give a brief mathematical explanation for each match.



iii) IS THE DERIVATIVE AT a) SINCE IT'S DERIVATIVE D.N.E. @ a AND b. WHERE f HAS CORNERS

i) IS THE DERIVATIVE OF b) SINCE $f'(x) > 0$ ON (a,b) AND f IS INCREASING ON (a,b) AND $f'(x) < 0$ ON $(-\infty, a)$ AND (b, ∞) WHERE f IS DECREASING.

iv) IS THE DERIVATIVE OF c) SINCE $f'(x) > 0$ AND f IS ALWAYS INCREASING.

ii) IS THE DERIVATIVE OF d) SINCE $f'(x)$ DOES NOT EXIST AT a WHERE f HAS A CORNER.

Question 5. *(10 marks)* Do the first six steps of the seven step procedure for graphing the curve

$$f(x) = x^6 + 6x^5$$

You do not need to do the seventh step, that is, **you do not need to graph this curve.**

1) DOMAIN: $\mathbb{R}$

2) y-int: $x=0 \Rightarrow y = 0^6 + 6(0)^5 = 0 \quad \therefore (0,0)$

x-int: $y=0 \Rightarrow 0 = x^6 + 6x^5 = x^5(x+6)$

$\Rightarrow x = 0, -6$

$\therefore (0,0), \quad (-6,0)$

3) $\lim_{x \to \pm\infty} x^6 + 6x = \infty$ NO H.A.

4) NO V.A, SINCE f IS A POLYNOMIAL.

5) $f'(x) = 6x^5 + 30x^4 = 0$

$6x^4(x+5) = 0$

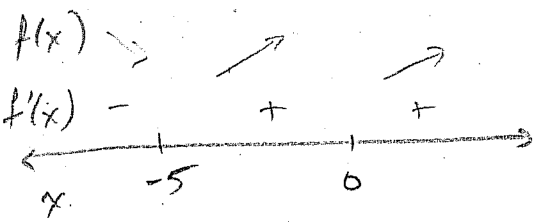
$\therefore x = 0, -5$

TEST POINTS:

$x = -6 \quad f'(-6) = -7776 < 0$

$x = -1 \quad f'(-1) = 24 > 0$

$x = 1 \quad f(1) = 1 > 0$



$\therefore f(-5) = -3125$ IS A RELATIVE MAXIMUM.

f IS DECREASING ON $(-\infty, -5)$

f IS INCREASING ON $(-5, 0)$ AND $(0, \infty)$.

6) $f''(x) = 30x^4 + 120x^3 = 0$

$30x^3(x+4) = 0$

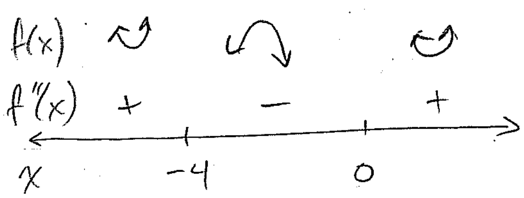
$\therefore x = 0, -4$

TEST POINTS.

$x = -5 \quad f''(-5) = 3750 > 0$

$x = -1 \quad f''(-1) = -90 < 0$

$x = 1 \quad f''(1) = 150 > 0$



$f(-4) = -2048 \qquad f(0) = 0$

$f'(-4) = 1536 \qquad f'(0) = 0$

$\therefore (-4, -2048)$ AND $(0,0)$ ARE INFLECTION POINTS.

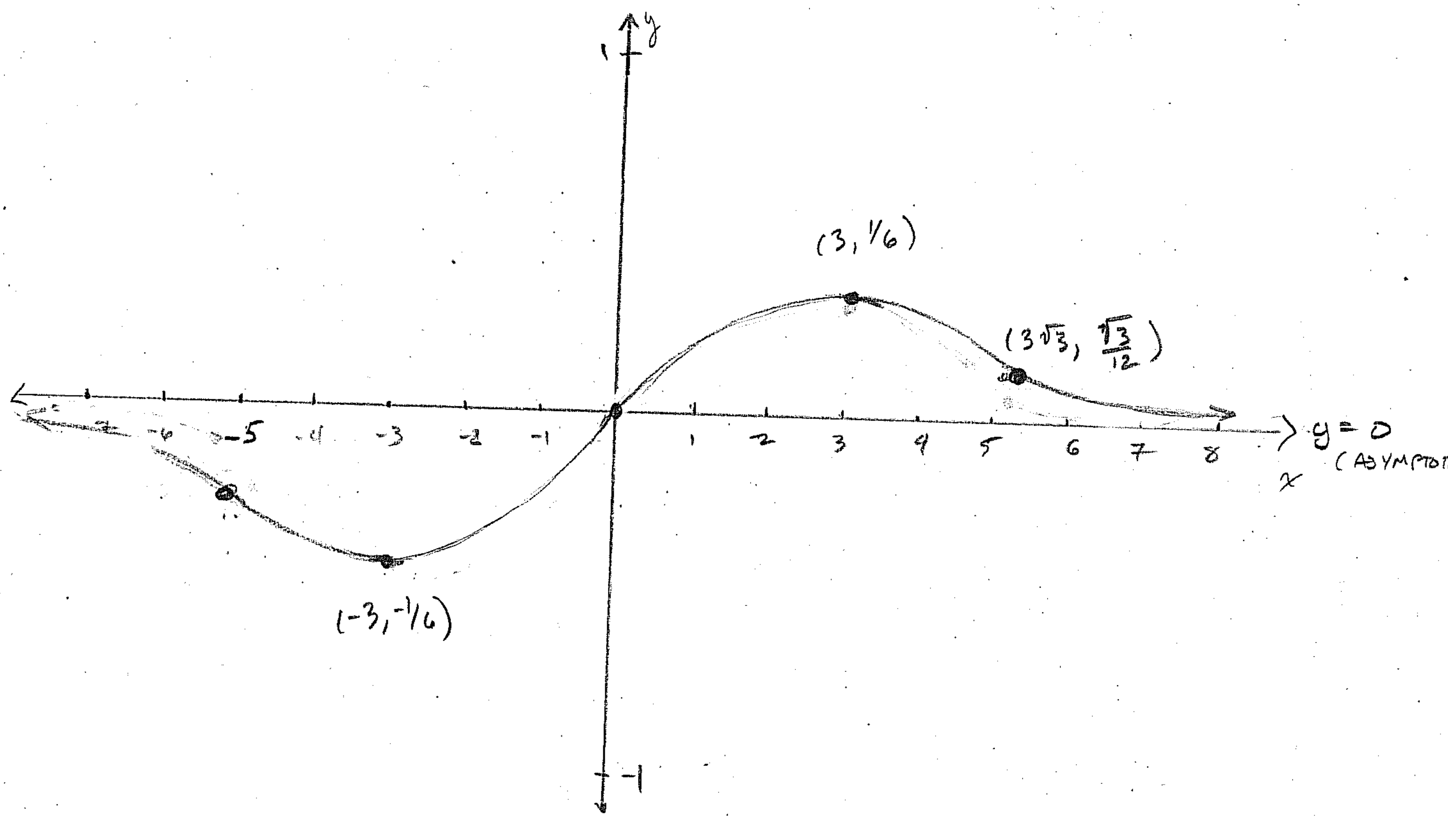
f IS CONCAVE UPWARD ON $(-\infty, -4)$ AND $(0, \infty)$

f IS CONCAVE DOWNWARD ON $(-4, 0)$

Question 6. *(5 marks.)* The function

$$f(x) = \frac{x}{x^2+9}$$

has the following properties:
Domain: $(-\infty,\infty)$. **x-intercept, y-intercept**: $(0,0)$.
Horizontal asymptote: $y = 0$. **Vertical asymptote:** None.
f **is increasing on:** $(-3,3)$. f **is decreasing on:** $(-\infty,-3)$ and $(+3,\infty)$.
Relative minimum: $(-3,-1/6)$. **Relative maximum:** $(3,1/6)$
Concave upward on: $(-3\sqrt{3},0)$ and $(3\sqrt{3},\infty)$.
Concave downward on: $(-\infty,-3\sqrt{3})$ and $(0,3\sqrt{3})$.
Inflection Points: $(3\sqrt{3},\sqrt{3}/12)$ and $(-3\sqrt{3},-\sqrt{3}/12)$ Sketch the graph of $f(x)$.
AND $(0,0)$



Question 7. *(5 marks.)* Use the second derivative test to find the local extrema for:

$$f(x) = x + \frac{4}{x}$$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2}$$

$f'(x) = 0$

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = \pm 2$

$f'(x)$ D.N.E

$x = 0$

BUT $f(0)$ IS UNDEFINED

SO NO RELATIVE MAX OR MIN AT $x=0$.

$$f''(x) = \frac{8}{x^3}$$

$f''(2) = \frac{8}{(2)^3} = 1 > 0$ $\therefore$ f IS CONCAVE UP AT $x=2$

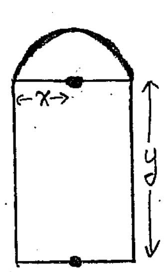
$\therefore$ $f(2) = 2 + \frac{4}{2} = 4$ IS A RELATIVE MINIMUM

$f''(-2) = \frac{8}{(-2)^3} = -1 < 0$ $\therefore$ f IS CONCAVE DOWN AT $x = -2$

$f(-2) = -2 + \frac{4}{-2} = -4$ IS A RELATIVE MAXIMUM.

Question 8. *(8 marks)* A window has the shape of a semicircle above a rectangle. If the window is to have a outer perimeter of 26ft, what should its dimensions be to allow the maximum amount of light through the window?

1.



2. $A = \frac{1}{2}\pi x^2 + 2xy$

3. $\frac{1}{2}(2\pi x) + 2y + x \Rightarrow P = 26$

$\pi x + 2y + 2x = 26$

$2y = 26 - 2x - \pi x$

$y = 13 - x - \frac{\pi}{2} x$

$\therefore A = \frac{1}{2}\pi x^2 + 2x\left(13 - x - \frac{\pi}{2}x\right)$

$= \frac{\pi}{2}x^2 + 26x - 2x^2 - \pi x^2 = 26x - \left(2 + \frac{\pi}{2}\right)x^2 = f(x)$

RESTRICTIONS:

$x > 0$

$2x + \frac{1}{2}(2\pi x) \leq 26$

$2x + \pi x \leq 26$

$x(2+\pi) \leq 26$

$x \leq \frac{26}{2+\pi}$

$\therefore$ THE DOMAIN OF f IS $\left[0, \frac{26}{2+\pi}\right]$

$f'(x) = 26 - 2\left(2 + \frac{\pi}{2}\right)x = 0$

$26 = (4+\pi)x$

$\frac{26}{4+\pi} = x$

END POINTS

$f(0) = 0$

$f\left(\frac{26}{2+\pi}\right) = 40.167$

C.N.

$f\left(\frac{26}{4+\pi}\right) = 47.328$

$\therefore x = \frac{26}{4+\pi} \approx 3.64\text{ft}$

$y = 13 - \left(\frac{26}{4+\pi}\right) - \frac{\pi}{2}\left(\frac{26}{4+\pi}\right)$

≈ 3.64