Dawson College: Calculus I: 201-NYA-05 15

November 26, 2010

Last Name: SOLUTIONS

First Name: __________

Student ID: __________

Test 3

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to use **correct notation**.

Question 1. *(5 marks)*
(a) Find $\frac{dy}{dx}$ for $(x-y^2)^6 + xy = 1$.
(b) Find the tangent line to this curve at the point $(1,1)$.

a) $\frac{d}{dx}(x-y^2)^6 + \frac{d}{dx}(xy) = \frac{d}{dx}(1)$

$6(x-y^2)^5(1-2yy') + (1y + xy') = 0$

$6(x-y^2)^5 - 12yy'(x-y^2)^5 + y + xy' = 0$

$xy' - 12yy'(x-y^2)^5 = -6(x-y^2)^5 - y$

$y'[x - 12y(x-y^2)^5] = -6(x-y^2)^5 - y$

$$\frac{dy}{dx} = y' = \frac{-6(x-y^2)^5 - y}{x - 12y(x-y^2)^5}$$

b) $m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \frac{-6(1-1)^5 - 1}{1 - 12(1)(1-1)^5} = -1$

$y = mx + b$

$1 = (-1)(1) + b$

$2 = b$

$\therefore \boxed{y = -x + 2}$

Question 2. *(8 marks)* Suppose the demand equation for a certain product is

$$210x^2 + 11P^2 = 4300$$

where x represents the number of units demanded each week when the unit price is $\$p$. How much is the quantity demanded increasing when the unit price is \$11 per unit and unit price is decreasing at a rate of \$0.21 per unit per week? (You can use decimals, round your final answer to the nearest unit.) X IS IN UNITS OF A THOUSAND

$p = 11$, $x = ?$ $\frac{dp}{dt} = -0.21$, $\frac{dx}{dt} = ?$

$$210x^2 + 11(11)^2 = 4300$$
$$210x^2 = 2969$$
$$x^2 = 14.13810$$
$$x = 3.7601$$

$$\frac{d}{dt}(210x^2) + \frac{d}{dt}(11p^2) = \frac{d}{dt}(4300)$$

$$420x\frac{dx}{dt} + 22p\frac{dp}{dt} = 0$$

$$420(3.7601)\frac{dx}{dt} + 22(11)(-0.21) = 0$$

$$1579.228\frac{dx}{dt} = 50.82$$

$$\frac{dx}{dt} = 0.0322$$

∴ THE QUANTITY DEMANDED IS INCREASING BY 32 UNITS PER WEEK AT THE TIME UNDER CONSIDERATION

Question 3. *(6 marks)* Find the asymptotes of the following functions (you do not need to sketch the graphs).

$$\textbf{(a)} f(x) = \frac{-2x^2-4x+1}{x^2+7x+12}$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{-2-4/x+1/x^2}{1+7/x+12/x^2} = \frac{-2}{1} = -2$$

$\therefore$ $y=-2$ IS THE HORIZONTAL ASYMPTOTE

$x^2+7x+12=0 \Rightarrow (x+3)(x+4)=0 \Rightarrow x=-3,-4$

NOW $-2(-3)^2-4(-3)+1 \neq 0$

AND $-2(-4)^2-4(-2)^2+1 \neq 0$

$\therefore$ $x=-3$ AND $x=-4$ ARE VERTICAL ASYMPTOTES.

$$\textbf{(b)} g(x) = \frac{13}{x^3-x}$$

$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} \frac{13/x^3}{1-1/x^2} = \frac{0}{1} = 0$$

$\therefore$ $y=0$ IS THE HORIZONTAL ASYMPTOTE.

$x^3-x=0$

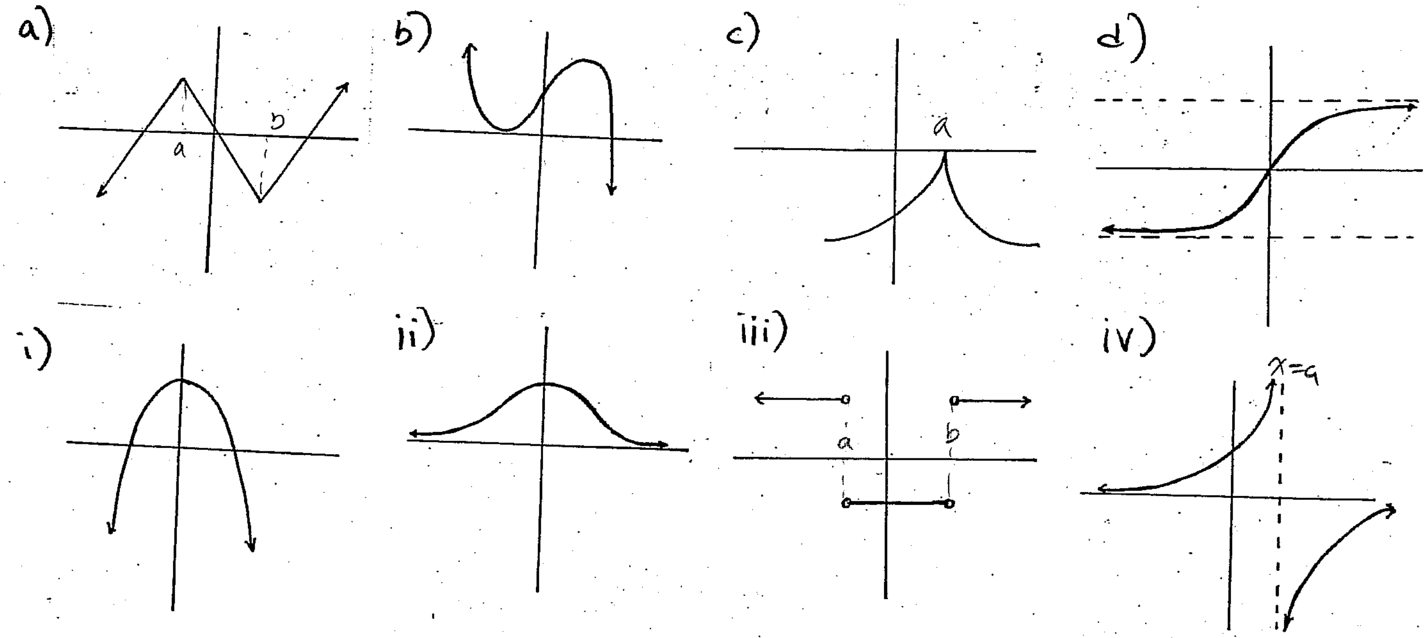
$x(x^2-1)=0$

$x(x+1)(x-1)=0$

$x=-1,0,1.$

$\therefore$ $x=-1$, $x=0$ AND $x=1$ ARE VERTICAL ASYMPTOTES.

Question 4. *(6 marks)* Match the graph of each function from a, b, c, d with the graph of its derivative from i, ii, iii, iv. Give a brief mathematical explanation for each match.



iii) IS THE DERIVATIVE OF a) SINCE $f'(x)$ D.N.E AT a AND b WHERE f HAS CORNERS.

i) IS THE DERIVATIVE OF b) BECAUSE $f'(x)>0$ WHERE f IS INCREASING AND $f'(x)<0$ WHERE f IS DECREASING.

iv) IS THE DERIVATIVE OF c) SINCE f HAS A CORNER AT a AND $f'(x)$ D.N.E AT a

ii) IS THE DERIVATIVE OF d) SINCE $f'(x)>0$ EVERYWHERE AND f IS INCREASING EVERYWHERE.

Question 5. *(10 marks)* Do the first six steps of the seven step procedure for graphing the curve

$$f(x) = x^6 - 6x^5$$

You do not need to do the seventh step, that is, **you do not need to graph this curve.**

1. DOMAIN: $\mathbb{R}$

2. y-int: $x=0 \Rightarrow y = 0^6 - 6(0)^5 = 0 \quad \therefore (0,0)$

x-int: $y=0 \Rightarrow 0 = x^6 - 6x^5$

$0 = x^5(x-6)$

$x = 0, 6 \quad \therefore (0,0), (6,0)$

3. $\lim\limits_{x \to \pm\infty} x^6 - 6x^5 = \infty \quad \therefore$ NO H.A.

4. NO. V.A. (POLYNOMIAL)

5. $f'(x) = 6x^5 - 30x^4 = 0$

$6x^4(x-5) = 0$

$x = 0, 5$

TEST POINTS:

$x = -1 \quad f'(-1) = -36 < 0$

$x = 1 \quad f'(1) = -24 < 0$

$x = 6 \quad f'(6) = 7776 > 0$

x		0		5	
$f'(x)$	$-$		$+$		$+$
$f(x)$	$\searrow$		$\searrow$		$\nearrow$

$\therefore$ f IS DECREASING ON $(-\infty, 0)$ AND $(0,5)$

f IS INCREASING ON $(5, \infty)$

$\therefore f(5) = -3125$ IS A RELATIVE MINIMUM.

6. $f''(x) = 30x^4 - 120x^3 = 0$

$30x^3(x-4) = 0$

$\therefore x = 0, 4$

TEST POINTS:

$x = -1 \quad f''(-1) = 150 > 0$

$x = 1 \quad f''(1) = -90 < 0$

$x = 5 \quad f''(5) = 3750 > 0$

x		0		4	
$f''(x)$	$+$		$-$		$+$
$f(x)$	$\cup$		$\cap$		$\cup$

f IS CONCAVE UPWARD ON $(-\infty, -4)$ AND $(0, \infty)$

f IS CONCAVE DOWNWARD ON $(0,4)$

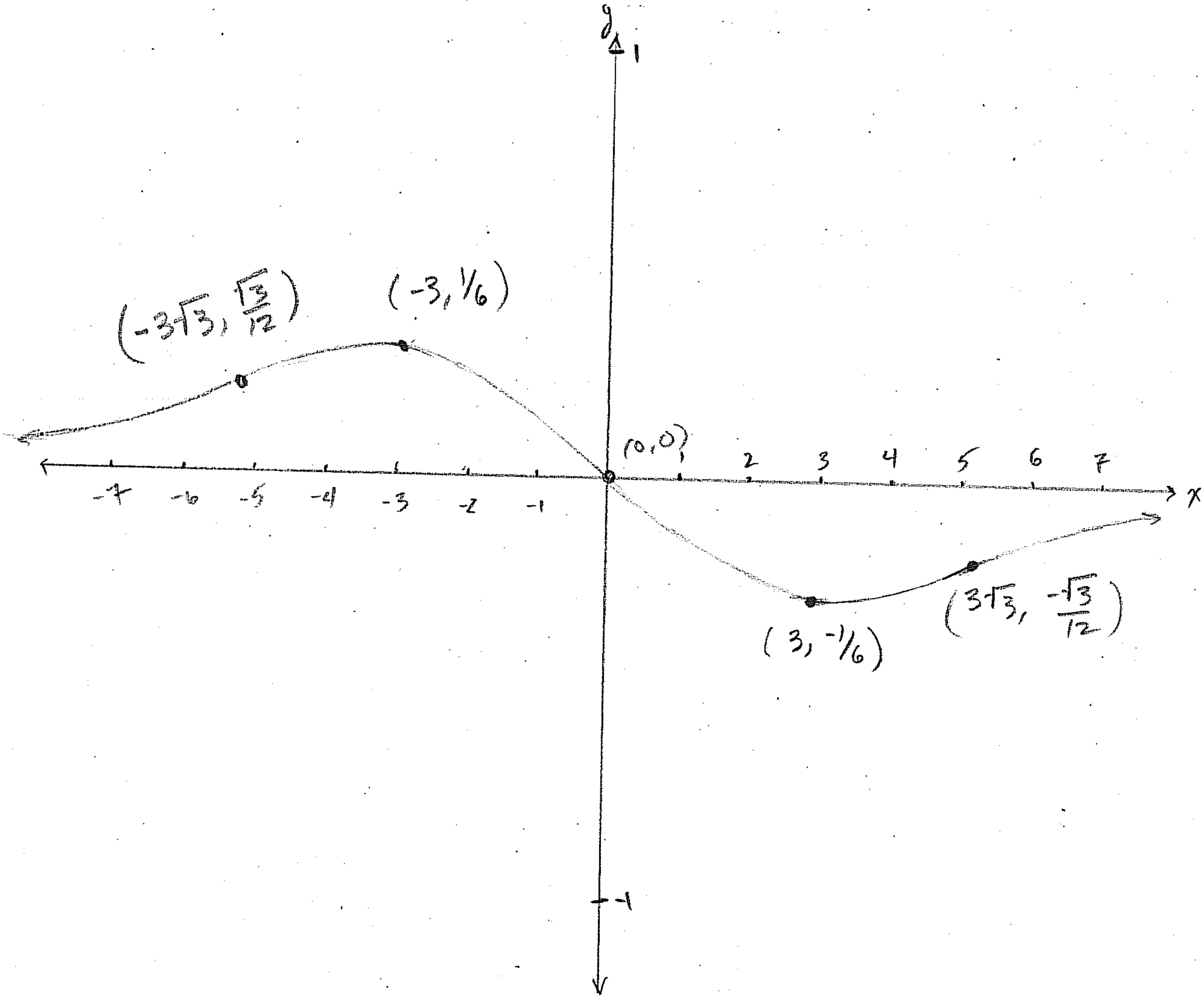
$f(0) = 0,\ f'(0) = 0 \Rightarrow (0,0)$ IS AN INFLECTION POINT

$f(4) = -2048,\ f'(4) = -1536 \Rightarrow (4, -2048)$ IS AN INFLECTION POINT.

Question 6. *(5 marks.)* The function

$$f(x) = \frac{-x}{x^2+9}$$

has the following properties:
Domain: $(-\infty, \infty)$. **x-intercept, y-intercept**: $(0,0)$.
Horizontal asymptote: $y = 0$. **Vertical asymptote:** None.
f **is decreasing on:** $(-3,3)$. f **is increasing on:** $(-\infty,-3)$ and $(+3,\infty)$.
Relative minimum: $(3,-1/6)$. **Relative maximum:** $(-3,1/6)$
Concave downward on: $(-3\sqrt{3},0)$ and $(3\sqrt{3},\infty)$.
Concave upward on: $(-\infty,-3\sqrt{3})$ and $(0,3\sqrt{3})$.
Inflection Points: $(3\sqrt{3},-\sqrt{3}/12)$ and $(-3\sqrt{3},+\sqrt{3}/12)$ Sketch the graph of $f(x)$.
AND $(0,0)$



Question 7. *(5 marks.)* Use the second derivative test to find the local extrema for:

$$f(x) = x + \frac{16}{x}$$

$f'(x) = 1 - \frac{16}{x^2}$

$= \frac{x^2 - 16}{x^2}$

$f'(x) = 0$

$\Rightarrow x^2 - 16 = 0$

$(x+4)(x-4) = 0$

$x = \pm 4$

$f'(x)$ D.N.E

$\Leftrightarrow x = 0$

BUT $f(0)$ IS NOT DEFINED

$\therefore$ NO EXTREMA AT $x = 0$.

$f''(x) = \frac{32}{x^3}$

$f''(-4) = \frac{32}{(-4)^3} = -\frac{1}{2} < 0 \Rightarrow$ f IS CONCAVE DOWN AT $x = -4$

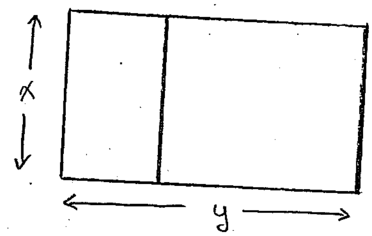
$\therefore f(-4) = -4 + \frac{16}{-4} = -8$ IS A RELATIVE MAXIMUM

$f''(4) = \frac{32}{4^3} = \frac{1}{2} > 0 \Rightarrow$ f IS CONCAVE UP AT $x = 4$.

$\therefore f(4) = 4 + \frac{16}{4} = 8$ IS A RELATIVE MINIMUM.

Question 8. *(8 marks)* A farmer wants to build a fence according to the diagram below. She has 500m of fencing with which to build the fence. Find the dimensions of the fence that yield the maximum area.

1.



2. $A = xy$

3. $P = 2y + 3x = 500 \Rightarrow y = 250 - \frac{3}{2}x$

$\therefore A = x\left(250 - \frac{3}{2}x\right)$

$= 250x - \frac{3}{2}x^2 = f(x)$

RESTRICTIONS!

$x \geq 0$

$3x \leq 500 \Rightarrow x \leq \frac{500}{3}$

$\therefore$ THE DOMAIN OF f IS $\left[0, \frac{500}{3}\right]$

4. $f'(x) = 250 - 3x = 0$

$250 = 3x$

$\frac{250}{3} = x$

END POINTS

$\therefore f(0) = 0$

$f\left(\frac{500}{3}\right) = 250\left(\frac{500}{3}\right) - \frac{3}{2}\left(\frac{500}{3}\right)^2 = 0$

C.N

$f\left(\frac{250}{3}\right) = 250\left(\frac{250}{3}\right) - \frac{3}{2}\left(\frac{250}{3}\right)^2 = \frac{250^2}{3} - \frac{250^2}{6}$

$= \frac{250^2}{6} = \frac{31250}{3}$ ← MAXIMUM.

$\therefore x = \frac{250}{3}m$, $y = 250 - \frac{3}{2}\left(\frac{250}{3}\right) = 125m$ YIELDS THE MAXIMUM AREA.