Dawson College: **Calculus II**: 201-NYB-05-S03: Fall 2010

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Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate using the definition of the definite integral

$$\int_1^3 -x^2-x+1\,dx. = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{where } f(x) = -x^2-x+1$$

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[-\left(1+\frac{2i}{n}\right)^2 - \left(1+\frac{2i}{n}\right)+1\right]\frac{2}{n} \qquad x_i = a + i\Delta x = 1+\frac{2i}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n}\left[-\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right) - \left(1+\frac{2i}{n}\right)+1\right]\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n}\left[-1-\frac{4i}{n}-\frac{4i^2}{n^2} \cancel{-1} - \frac{2i}{n} \cancel{+1}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n}\left[-1-\frac{6i}{n}-\frac{4i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\sum_{i=1}^{n} -1 - \frac{6}{n}\sum_{i=1}^{n} i - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[-n - \frac{\cancel{6}^{3}}{n}\,\frac{n(n+1)}{\cancel{2}} - \frac{\cancel{4}^{2}}{n^2}\,\frac{n(n+1)(2n+1)}{\cancel{6}\,3}\right]$$

$$= \lim_{n\to\infty}\left[-\frac{2n}{n} - \frac{6n.(n+1)}{n\;\;n} - 4\,\frac{n}{n}\cdot\frac{(n+1)}{n}\cdot\frac{(2n+1)}{3n}\right]$$

$$= -2 - 6\cdot 1 - 4\cdot 1\cdot 1\cdot\frac{2}{3}$$

$$= -2 - 6 - \frac{8}{3}$$

$$= \frac{-32}{3}$$

Question 2. *(2 marks)* Evaluate the indefinite integral:

$$\int \tan x - \csc x + \frac{1}{\sqrt{11-x^2}}\,dx = -\ln|\cos x| + \ln|\csc x + \cot x| + \arcsin\left(\frac{x}{\sqrt{11}}\right) + C$$

Question 3. *(4 marks)* Evaluate the definite integral:

$$\int_1^e \frac{(x-1)(x+1)^2}{x} + \frac{2}{x}\,dx = \int_1^e \frac{(x^2-1)(x+1)}{x} + \frac{2}{x}\,dx$$

$$= \int_1^e \frac{x^3 - x + x^2 - 1}{x} + \frac{2}{x}\,dx$$

$$= \int_1^e x^2 - 1 + x - \frac{1}{x} + \frac{2}{x}\,dx$$

$$= \int_1^e x^2 + x - 1 + \frac{1}{x}\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} - x + \ln|x|\right]_1^e$$

$$= \frac{e^3}{3} + \frac{e^2}{2} - e + \ln|e| - \left[\frac{1}{3} + \frac{1}{2} - 1 + \ln 1\right]$$

$$= \frac{e^3}{3} + \frac{e^2}{2} - e + 1 - \left[\frac{1}{3} + \frac{1}{2} - 1\right]$$

$$= \frac{e^3}{3} + \frac{e^2}{2} - e + \frac{7}{6}$$

Question 4. *(4 marks)* Evaluate the definite integral:

$$\int_{\pi/4}^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\pi/3} \frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta$$

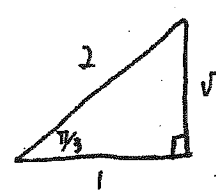
$$= \int_{\pi/4}^{\pi/3} \frac{\sin\theta\,\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$$

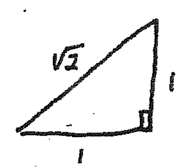
$$= \Big[-\cos\theta\Big]_{\pi/4}^{\pi/3}$$

$$= -\cos\frac{\pi}{3} + \cos\frac{\pi}{4}$$

$$= -\frac{1}{2} + \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2}$$





Question 4. *(5 marks)* A super galactic space ship is traveling towards Pluto at a speed of

$$v(t) = \frac{1}{\pi}\frac{1}{t^2+16}$$

Terameters per hour. What distance was traveled after 4 hours? What is the average speed of the ship after 4 hours of travel?

$$\text{distance} = \int_0^4 \frac{1}{\pi}\cdot\frac{1}{t^2+16}\,dt$$

$$= \frac{1}{\pi}\int_0^4 \frac{1}{t^2+16}\,dt$$

$$= \frac{1}{\pi}\left[\frac{1}{4}\arctan\frac{t}{4}\right]_0^4$$

$$= \frac{1}{4\pi}\left[\arctan\frac{4}{4} - \arctan\frac{0}{4}\right]$$

$$= \frac{1}{4\pi}\left[\frac{\pi}{4}\right] = \frac{1}{16} \text{ Tera meters}$$

$$\text{Avg speed} = \frac{1}{b-a}\int_a^b v(t)\,dt$$

$$= \frac{1}{4-0}\int_0^4 v(t)\,dt$$

$$= \frac{1}{4}\cdot\frac{1}{16}$$

$$= \frac{1}{64} \text{ Tera meters / hour.}$$

Question 5. *(5 marks + 1 bonus mark to simplify completely)* Compute the derivative:

$$\frac{d}{dx}\Bigg[\underbrace{\int_{\arcsin 3x}^{\arctan 2x} \sin^{100} t \tan^{1001} t\, dt}_{h(x)}\Bigg]$$

$$h(x) = \int_{\arcsin 3x}^{0} \sin^{100} t \tan^{1001} t\, dt + \int_0^{\arctan 2x} \sin^{100} t \tan^{1001} t\, dt$$

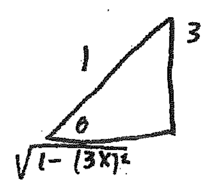
$$= -\int_0^{\arcsin 3x} \sin^{100} t \tan^{1001} t\, dt + \int_0^{\arctan 2x} \sin^{100} t \tan^{1001} t\, dt$$

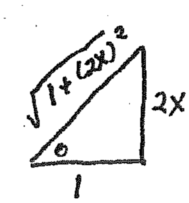
$$= -f(g_1(x)) + f(g_2(x))$$

where $f(x) = \int_0^x \sin^{100} t \tan^{1001} t\, dt \Rightarrow f'(x) = \sin^{100} x \tan^{1001} x$ by 2^{nd} FTC

$$g_1(x) = \arcsin 3x \Rightarrow g_1'(x) = \frac{1}{\sqrt{1-(3x)^2}}\cdot 3$$

$$g_2(x) = \arctan 2x \Rightarrow g_2'(x) = \frac{1}{1+(2x)^2}\cdot 2$$





$$h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$$

$$= -\sin^{100}(\arcsin 3x)\tan^{1001}(\arcsin 3x)\cdot\frac{3}{\sqrt{1-(3x)^2}} + \sin^{100}(\arctan(2x))\tan^{1001}(\arctan 2x)\cdot\left(\frac{2}{1+(2x)^2}\right)$$

$$= -(3x)^{100}\cdot\left(\frac{3x}{\sqrt{1-(3x)^2}}\right)^{1001}\cdot\frac{3}{\sqrt{1-(3x)^2}} + \left(\frac{2x}{\sqrt{1+(2x)^2}}\right)^{100}(2x)^{1001}\cdot\frac{2}{1+(2x)^2}$$

$$= \frac{-3\cdot(3x)^{1101}}{(1-(3x)^2)^{501}} + \frac{2\cdot(2x)^{1101}}{(1+(2x)^2)^{1051}}$$

Question 6. *(4 marks)* Evaluate the indefinite integral:

$\int x\sqrt[3]{x-1}\,dx \overset{①②③}{=} \int (u+1)\sqrt[3]{u}\,du = \int u^{4/3} + u^{1/3}\,du$

$u = x-1$ ①

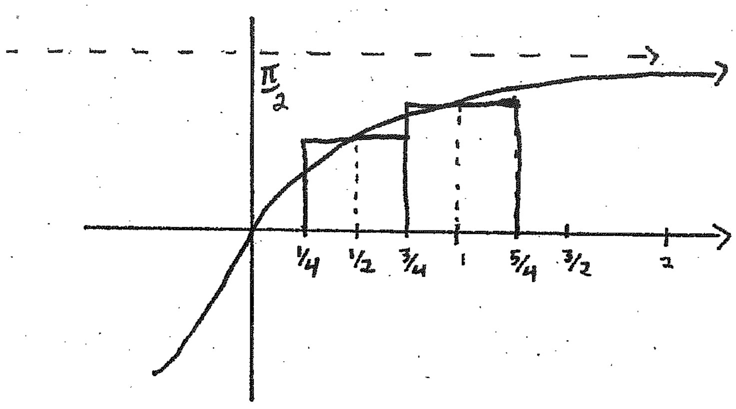
$du = dx$ ②

$x = u+1$ ③

$= \frac{3u^{7/3}}{7} + \frac{3u^{4/3}}{4} + C$

$\overset{①}{=} \frac{3}{7}(x-1)^{7/3} + \frac{3}{4}(x-1)^{4/3} + C$

Question 7. *(4 marks)* Estimate the area under the graph of $f(x) = \arctan x$ from $x = \frac{1}{4}$ to $x = \frac{5}{4}$ using two rectangles and the Midpoint Rule. Sketch the curve and approximating rectangles.



$n = 2, \quad \Delta x = \frac{b-a}{n} = \frac{\frac{5}{4} - \frac{1}{4}}{2} = \frac{1}{2}$

$x_i = a + i\Delta x = \frac{1}{4} + \frac{2i}{4}$

$x_0 = \frac{1}{4}$ $\quad x_1^* = \frac{1/4 + 3/4}{2} = \frac{1}{2}$

$x_1 = \frac{3}{4}$ $\quad x_2^* = \frac{3/4 + 5/4}{2} = 1$

$x_2 = 5/4$

Area $\approx R_1 + R_2$

$= f(x_1^*)\Delta x + f(x_2^*)\Delta x$

$= \arctan\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right) + \arctan(1)\left(\frac{1}{2}\right)$

$= \frac{1}{2}\arctan\left(\frac{1}{2}\right) + \frac{1}{2}\cdot\frac{\pi}{4}$

$= \frac{\pi}{8} + \frac{1}{2}\arctan\left(\frac{1}{2}\right)$

Question 8. *(4 marks)* Evaluate the indefinite integral:

$$\int \frac{1}{e^{\tan z}\cos^2 z}\,dz = \int e^{-\tan z}\sec^2 z\,dz \overset{①,②}{=} \int e^{u}\,(-du)$$

$$= -\int e^{u}\,du$$

$$= -e^{u} + C$$

$$= -e^{-\tan z} + C$$

$$u \overset{①}{=} -\tan z$$

$$du = -\sec^2 z\,dz$$

$$-du \overset{②}{=} \sec^2 z\,dz$$

Question 9. *(3 marks)* Prove (without using the Fundamental Theorem of Calculus):

$$\int_a^b c\,dx = c(b-a)$$

where c is a constant.

$$\int_a^b c\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad \Delta x = \frac{b-a}{n} \qquad f(x) = c$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} c\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} c$$

$$= \lim_{n\to\infty}\left(\frac{b-a}{\cancel{n}}\right) c\cancel{n}$$

$$= c(b-a)$$

Bonus Question. *(3 marks)*
Evaluate the following limit:

Let $f(x) = \int_{\pi}^{\arctan x} \tan z \sin^{101} z \, dz$

$$\lim_{\zeta \to 0} \frac{\int_{\pi}^{\arctan(x+\zeta)} \tan z \sin^{101} z \, dz - \int_{\pi}^{\arctan x} \tan z \sin^{101} z \, dz}{\zeta} = f'(x)$$

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So $f(x) = k(g(x))$ where $k(x) = \int_{\pi}^{x} \tan z \sin^{101} z \, dz$

$\Rightarrow K'(x) = \tan x \sin^{101} x$ by 2^{nd} FTC

and $g(x) = \arctan x \Rightarrow g'(x) = \frac{1}{1+x^2}$

$$\therefore f'(x) = K'(g(x))\, g'(x)$$

$$= \tan(\arctan x) \sin^{101}(\arctan x) \cdot \frac{1}{1+x^2}$$

$$= \frac{x \sin^{101}(\arctan x)}{1+x^2}$$