Dawson College: Calculus II: 201-NYB-05-S03: Fall 2010

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Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Evaluate [redacted] the definite integral

$$\int_0^{1/2} \arcsin x \, dx. = \Big[uv\Big]_0^{1/2} - \int_0^{1/2} v\,du$$

$u = \arcsin x$ $du = \frac{1}{\sqrt{1-x^2}}\,dx$

$v = x$ $dv = dx$

$$= \Big[x \arcsin x\Big]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}\arcsin\frac{1}{2} - 0\arcsin 0 - \int_1^{3/4} \frac{1}{\sqrt{u}}\left(\frac{du}{-2}\right)$$

$u = 1 - x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

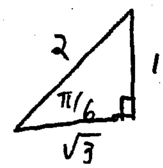
$u(0) = 1$

$u(\frac{1}{2}) = 3/4$

$$= \frac{1}{2}\arcsin\frac{1}{2} + \frac{1}{2}\Big[2\sqrt{u}\Big]_1^{3/4}$$

$$= \frac{1}{2}\frac{\pi}{6} + \sqrt{\frac{3}{4}} - \sqrt{1}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$



Question 2. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{t^2-3t-5}{t^3+5t}\,dt = \int \frac{t^2-3t-5}{t(t^2+5)}\,dt$$

$$\frac{t^2-3t-5}{t(t^2+5)} = \frac{A}{t} + \frac{Bt+C}{t^2+5}$$

$$t^2-3t-5 = A(t^5+5) + (Bt+C)t$$

$t=0:$

$$(0)^2-3(0)-5 = A(0^2+5) + (B(0)+C)(0)$$

$$-5 = 5A$$

$$-1 = A$$

$t=1:$

$$(1)^2-3(1)-5 = (-1)((1)^2+5) + (B(1)+C)(1)$$

$$-7 = -6 + B + C$$

$$-1 = B + C \quad (1)$$

$t=-1:$

$$(-1)^2-3(-1)-5 = (-1)((-1)^2+5) + (B(-1)+C)(-1)$$

$$= -6 + B - C$$

$$5 = B - C \quad (2)$$

$(1)+(2):$

$$4 = 2B$$

$$2 = B$$

$$\therefore\ C = -3$$

$$\int \frac{t^2-3t-5}{t^3+5t}\,dt = \int \frac{-1}{t} + \frac{2t-3}{t^2+5}\,dt$$

$$= -\ln|t| + \int \frac{2t}{t^2+5}\,dt - 3\int \frac{1}{t^2+5}\,dt$$

$u = t^2+5$

$du = 2t\,dt$

$$= -\ln|t| + \int \frac{1}{u}\,du - \frac{3}{\sqrt{5}}\arctan\frac{t}{\sqrt{5}} + C$$

$$= -\ln|t| + \ln|u| - \frac{3}{\sqrt{5}}\arctan\frac{t}{\sqrt{5}} + C$$

$$= -\ln|t| + \ln(t^2+5) - \frac{3}{\sqrt{5}}\arctan\frac{t}{\sqrt{5}} + C$$

Question 3. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{1}{\sqrt{x^2+16}}\,dx = \int \frac{1}{\sqrt{(4\tan\theta)^2+16}}\,4\sec^2\theta\,d\theta$$

$x = 4\tan\theta$

$dx = 4\sec^2\theta\,d\theta$

$$= \int \frac{1}{\sqrt{16\tan^2\theta+16}}\,4\sec^2\theta\,d\theta$$

$$= \int \frac{1}{\sqrt{16(\tan^2\theta+1)}}\,4\sec^2\theta\,d\theta$$

$$= \int \frac{1}{\sqrt{16\sec^2\theta}}\,4\sec^2\theta\,d\theta$$

$$= \int \frac{1}{\cancel{4\sec\theta}}\,4\sec^2\theta\,d\theta$$

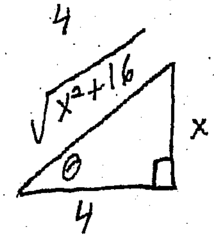
$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

<u>note:</u> $x = 4\tan\theta$

$\frac{x}{4} = \tan\theta$



$\sec\theta = \frac{hyp}{adj}$

$= \frac{\sqrt{x^2+16}}{4}$

Question 4. *(5 marks)* Evaluate the indefinite integral:

$$\int \cot^3(3x)\csc^4(3x)\,dx = \int \cot^3(3x)\csc^2(3x)\csc^2(3x)\,dx$$

$$= \int \cot^3(3x)\left[\cot^2(3x)+1\right]\csc^2(3x)\,dx$$

$u = \cot(3x)$

$du = -\csc^2(3x)(3)\,dx$

$\frac{du}{-3} = \csc^2(3x)\,dx$

$$= \int u^3\left[u^2+1\right]\frac{du}{-3} = \frac{-1}{3}\int u^5+u^3\,du$$

$$= \frac{-1}{3}\left[\frac{u^6}{6}+\frac{u^4}{4}\right]+C$$

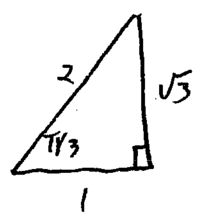
$$= \frac{-\cot^6(3x)}{18} - \frac{\cot^4(3x)}{12} + C$$

Question 5. *(5 marks)* Evaluate the definite integral:

$$\int_{\pi/4}^{\pi/6} \sin^2\theta\,d\theta = \int_{\frac{\pi}{4}}^{\pi/6} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/6} 1-\cos 2\theta\,d\theta$$

$$= \frac{1}{2}\left[\theta \quad \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/6}$$



$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{\sin \pi/3}{2}\right] - \frac{1}{2}\left[\frac{\pi}{4} - \frac{\sin \pi/2}{2}\right]$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}/2}{4} - \frac{\pi}{8} + \frac{1}{4}$$

$$= \frac{-\pi}{24} + \frac{2-\sqrt{3}}{8}$$

Question 6. (5 marks) Evaluate the limit:

$$\lim_{x\to\infty}\left(\arctan\frac{1}{x}\right)^{\frac{1}{x}} \qquad \text{I.F. } 0^0$$

Let $y = \lim_{x\to\infty}\left(\arctan\frac{1}{x}\right)^{1/x}$

$\ln y = \ln \lim_{x\to\infty}\left(\arctan\frac{1}{x}\right)^{1/x}$

$\ln y = \lim_{x\to\infty} \ln\left(\arctan\frac{1}{x}\right)^{1/x}$

$\ln y = \lim_{x\to\infty} \frac{1}{x}\ln\left(\arctan\frac{1}{x}\right) \qquad \text{I.F. } 0\cdot-\infty$

$\ln y = \lim_{x\to\infty} \frac{\ln\left(\arctan\frac{1}{x}\right)}{x} \qquad \text{I.F. } \frac{-\infty}{\infty}$

$\ln y = \lim_{x\to\infty} \frac{\frac{1}{\arctan(\frac{1}{x})}\cdot\frac{1}{1+(\frac{1}{x})^2}\cdot\frac{-1}{x^2}}{1}$

$\ln y = \lim_{x\to\infty} \frac{-1}{\arctan(\frac{1}{x})}\cdot\frac{1}{x^2+1} \qquad \text{I.F. } -\infty\cdot 0$

$\ln y = \lim_{x\to\infty} \frac{-1/x^2+1}{\arctan(\frac{1}{x})} \qquad \text{I.F. } \frac{0}{0}$

$\ln y = \lim_{x\to\infty} \frac{\frac{1}{(x^2+1)^2}\cdot 2x}{\frac{1}{1+(\frac{1}{x})^2}\cdot\frac{-1}{x^2}} \qquad \text{by } H$

$\ln y = \lim_{x\to\infty} \frac{\frac{2x}{(x^2+1)^2}}{\frac{-1}{x^2+1}}$

$\ln y = \lim_{x\to\infty} \frac{-2x(x^2+1)}{(x^2+1)^2} \qquad \therefore \lim_{x\to\infty}\left(\arctan\frac{1}{x}\right)^{1/x} = 1$

$\ln y = 0$

$y = e^0$

$y = 1$

Question 7. *(5 marks)* Determine whether the integral is convergent or divergent. Evaluate if convergent.

$$\int_{-\infty}^{0} xe^{3x}\,dx = \lim_{a \to -\infty} \int_a^0 xe^{3x}\,dx$$

$u = x \qquad du = dx$

$v = \frac{e^{3x}}{3} \qquad dv = e^{3x}\,dx$

$$= \lim_{a \to -\infty} \left[\left[uv \right]_a^0 - \int_a^0 v\,du \right]$$

$$= \lim_{a \to -\infty} \left[\left[\frac{xe^{3x}}{3} \right]_a^0 - \int_a^0 \frac{e^{3x}}{3}\,dx \right]$$

$$= \lim_{a \to -\infty} \left[\frac{0e^0}{3} - \frac{ae^{3a}}{3} - \left[\frac{e^{3x}}{9} \right]_a^0 \right]$$

$$= \lim_{a \to -\infty} \left[\underbrace{\frac{-ae^{3a}}{3}}_{\substack{\text{I.F.} \\ \infty \cdot 0}} - \frac{e^{3(0)}}{9} + \cancel{\frac{e^{3a}}{9}}^{\to 0} \right]$$

$$= -\frac{1}{9} - \lim_{a \to -\infty} \frac{a}{3e^{-3a}} \quad \text{I.F.} \quad \frac{-\infty}{\infty}$$

$$= -\frac{1}{9} - \lim_{a \to -\infty} \cancel{\frac{1}{-9e^{-3a}}}^{\to 0}$$

$$= -\frac{1}{9}$$

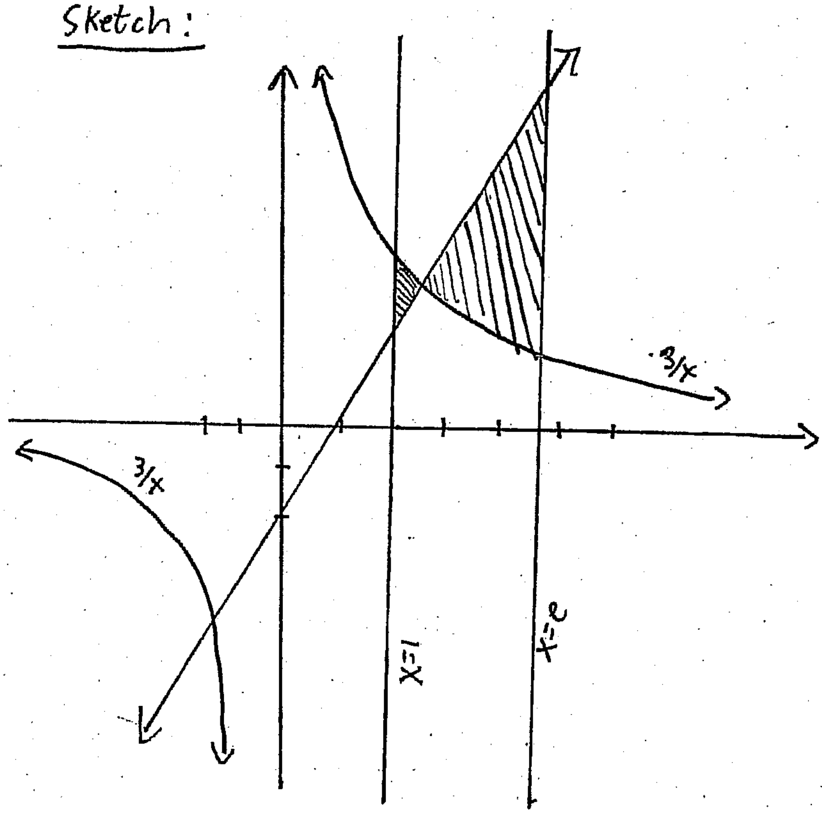
Question 8. *(5 marks)* Sketch the region enclosed by the given curves. Then find the area of the region.

$y = \frac{3}{x}$, $y = 2x - 1$, $x = 1$, $x = e$.

Intersection of $y = \frac{3}{x}$, $y = 2x-1$:

$$2x - 1 = \frac{3}{x}$$
$$(2x-1)x = 3$$
$$2x^2 - x = 3$$
$$2x^2 - x - 3 = 0$$
$$(2x-3)(x+1) = 0$$
$$x = \frac{3}{2} \qquad x = -1$$

Sketch :



$$\text{Area} = \int_1^{3/2} \frac{3}{x} - (2x-1)\,dx + \int_{3/2}^{e} (2x-1) - \frac{3}{x}\,dx$$

$$= \Big[3\ln|x| - x^2 + x\Big]_1^{3/2} + \Big[x^2 - x - 3\ln|x|\Big]_{3/2}^{e}$$

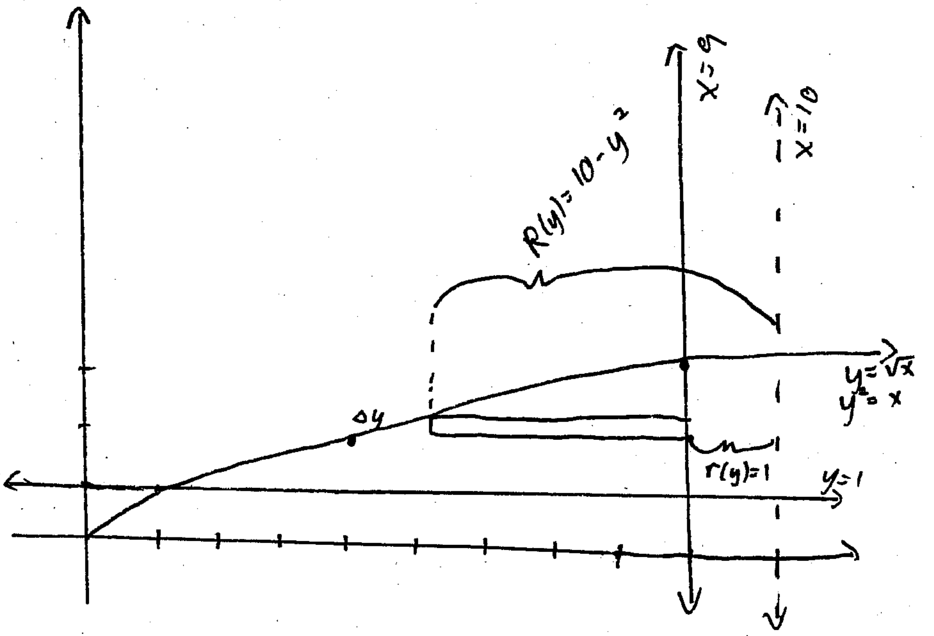
$$= 3\ln\tfrac{3}{2} - \left(\tfrac{3}{2}\right)^2 + \tfrac{3}{2} - 3\ln 1 - 1^2 + 1$$
$$+ e^2 - e - 3\ln e - \left(\tfrac{3}{2}\right)^2 + \tfrac{3}{2} + 3\ln\left(\tfrac{3}{2}\right)$$

$$= 6\ln\tfrac{3}{2} - \tfrac{9}{2} + 3 + e^2 - e - 3$$

$$= 6\ln\tfrac{3}{2} + e^2 - e - \tfrac{9}{2}$$

Question 9. *(5 marks)* Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y = \sqrt{x}$, $y = 1$, $x = 9$ about the x=10



rep. element:

$$\Delta V = \pi\left[(R(y))^2 - (r(y))^2\right]\Delta y$$
$$= \pi\left[(10-y^2)^2 - 1^2\right]\Delta y$$
$$= \pi\left[100 - 20y^2 + y^4 - 1\right]$$
$$= \pi\left[99 - 20y^2 + y^4\right]$$

$$V = \int_1^3 \pi\left[99 - 20y^2 + y^4\right]dy$$

$$= \pi\left[99y - \frac{20y^3}{3} + \frac{y^5}{5}\right]_1^3$$

$$= \pi\left[99(3) - \frac{20}{3}3^3 + \frac{3^5}{5}\right] - \pi\left[99 - \frac{20}{3} + \frac{1}{5}\right]$$

$$= \frac{412}{15}\pi$$

Bonus Question.
Given the following definite integral:

$$\int_1^2 x\ln x\,dx$$

a. *(3 marks)* For

$$\int_a^b f(x)\,dx$$

the error involved in using Simpson's Rule is

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where $|f^{(4)}(x)| \leq K$, $\forall x \in [a,b]$. How large do we have to choose n so that the approximation using Simpson's Rule is accurate to within 0.0001?

b. *(2 marks)* Using the n determined in part a. use Simpson's Rule to approximate the integral.

c. *(2 marks)* Compare the result obtained in part b. to the actual value. Discuss any conclusion that arise from part a. and b.

a) Lets determine K.

$f(x) = x\ln x$

$f'(x) = \ln x + \cancel{x}\frac{1}{\cancel{x}} = 1 + \ln x$

$f''(x) = \frac{1}{x}$

$f'''(x) = \frac{-1}{x^2}$

$f^{(iv)}(x) = \frac{2}{x^3}$ $\therefore$ $|f^{(4)}(x)| = \frac{2}{x^3} \leq K = 2$ since $f^{(4)}(x)$ largest at $x=1$ in $[1,2]$

So

$$\frac{K(b-a)^5}{180n^4} \leq 0.0001$$

$$\frac{2(2-1)^5}{180n^4} \leq$$

$$\frac{2(1)^5}{180(0.0001)} \leq n^4$$

$$3.24 \leq n$$

$\therefore$ $n = 4$.

b)

$$\int_1^2 x\ln x\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_n)\right]$$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$ $\qquad x_i = a + i\Delta x = 1 + i\left(\frac{1}{4}\right)$

$x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{6}{4}$, $x_3 = \frac{7}{4}$, $x_4 = 2$

$$= \frac{1}{12}\left[f(1) + 4f\left(\tfrac{5}{4}\right) + 2f\left(\tfrac{6}{4}\right) + 4f\left(\tfrac{7}{4}\right) + f(2)\right]$$

$$= \frac{1}{12}\left[1\ln 1 + 4\left(\tfrac{5}{4}\right)\ln\left(\tfrac{5}{4}\right) + 2\left(\tfrac{6}{4}\right)\ln\left(\tfrac{6}{4}\right) + n\left(\tfrac{7}{4}\right) + 2\ln 2\right]$$

$$\doteq 0.636309$$

Bonus Question.
Given the following definite integral:

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b. *(2 marks)* Using the n determined in part a. use Simpson's Rule to approximate the integral.

c. *(2 marks)* Compare the result obtained in part b. to the actual value. Discuss any conclusion that arise from part a. and b.

c) $\int_1^2 x \ln x dx = [uv]_1^2 - \int_1^2 v du$ $u = \ln x$ $du = \frac{1}{x} dx$

$v = \frac{x^2}{2}$ $dv = x dx$

$= \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}} dv$

$= \frac{2^2 \ln 2}{2} - \frac{1^2 \ln 1}{2} - \left[\frac{x^2}{4}\right]_1^2$

$= 2\ln 2 - \frac{4}{4} + \frac{1}{4}$

$= \ln 4 - 1 + \frac{1}{4}$

$\doteq 0.636294$

Error $= |\text{actual} - \text{estimated}|$

$= |0.636294 - 0.636309|$

$= 0.000014$

$\therefore$ Like expected because of the choice of n the approx. is accurate to within 0.0001.