

## Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (5 marks) Evaluate ~~the definite integral~~ the definite integral

$$\int_{\sqrt{2}}^2 x \operatorname{arcsec} x \, dx = [uv]_{\sqrt{2}}^2 - \int_{\sqrt{2}}^2 v \, du$$

$$u = \operatorname{arcsec} x \quad du = \frac{1}{x\sqrt{x^2-1}} dx$$

$$v = \frac{x^2}{2}$$

$$dv = x \, dx$$

$$= \left[ \frac{x^2}{2} \operatorname{arcsec} x \right]_{\sqrt{2}}^2 - \int_{\sqrt{2}}^2 \frac{x^2}{2} \frac{1}{x\sqrt{x^2-1}} dx$$

$$= \frac{2^2}{2} \operatorname{arcsec} 2 - \frac{(\sqrt{2})^2}{2} \operatorname{arcsec} \sqrt{2} - \frac{1}{2} \int_{\sqrt{2}}^2 \frac{x}{\sqrt{x^2-1}} dx$$

$$u = x^2 - 1$$

$$du = 2x \, dx$$

$$\frac{du}{2} = x \, dx$$

$$u(\sqrt{2}) = (\sqrt{2})^2 - 1 = 1$$

$$u(2) = 2^2 - 1 = 3$$

$$= 2\left(\frac{\pi}{3}\right) - \frac{\pi}{4} - \frac{1}{2} \int_1^3 \frac{1}{\sqrt{u}} \frac{du}{2}$$

$$= \frac{8\pi}{12} - \frac{3\pi}{12} - \frac{1}{4} \left[ 2\sqrt{u} \right]_1^3$$

$$= \frac{5\pi}{12} - \frac{1}{2} \sqrt{3} + \frac{1}{2}$$

Question 2. (5 marks) Evaluate the indefinite integral:

$$\int \frac{-t^2 + 3t + 5}{t^3 + 5t} dt$$

$$\frac{-t^2 + 3t + 5}{t^3 + 5t} = \frac{-t^2 + 3t + 5}{t(t^2 + 5)} = \frac{A}{t} + \frac{Bt + C}{t^2 + 5}$$

$$-t^2 + 3t + 5 = A(t^2 + 5) + (Bt + C)t$$

$$t = 0:$$

$$-0^2 + 3(0) + 5 = A(0^2 + 5) + (B(0) + C)0$$

$$5 = 5A$$

$$A = 1$$

$$t = 1:$$

$$-1^2 + 3(1) + 5 = A(1^2 + 5) + (B(1) + C)(1)$$

$$7 = 6 + B + C$$

$$1 = B + C \quad (1)$$

$$t = -1:$$

$$-(-1)^2 + 3(-1) + 5 = A((-1)^2 + 5) + (B(-1) + C)(-1)$$

$$1 = 6 + B - C$$

$$-5 = B - C \quad (2)$$

$$(1) + (2)$$

$$-4 = 2B$$

$$-2 = B$$

$$\therefore C = 3$$

$$\int \frac{-t^2 + 3t + 5}{t^3 + 5t} dt = \int \frac{1}{t} + \frac{-2t + 3}{t^2 + 5} dt$$

$$= \int \frac{1}{t} dt - \int \frac{2t}{t^2 + 5} dt + 3 \int \frac{1}{t^2 + 5} dt$$

$$u = t^2 + 5 \\ du = 2t dt$$

$$= \ln|t| - \int \frac{1}{u} du + \frac{3}{\sqrt{5}} \arctan \frac{t}{\sqrt{5}} + C$$

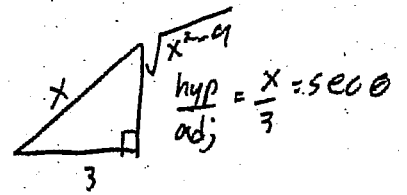
$$= \ln|t| - \ln|u| + \frac{3}{\sqrt{5}} \arctan \frac{t}{\sqrt{5}} + C$$

$$= \ln|t| - \ln(t^2 + 5) + \frac{3}{\sqrt{5}} \arctan \frac{t}{\sqrt{5}} + C$$

Question 3. (5 marks) Evaluate the indefinite integral:

$$\int \frac{1}{x^2 \sqrt{x^2 - 9}} dx \quad x = 3 \sec \theta \quad \Rightarrow$$

$$dx = 3 \sec \theta \tan \theta d\theta$$



$$= \int \frac{1}{(3 \sec \theta)^2 \sqrt{(3 \sec \theta)^2 - 9}} 3 \sec \theta \tan \theta d\theta$$

$$= \frac{1}{3} \int \frac{1}{\sec \theta \sqrt{9 \sec^2 \theta - 9}} \tan \theta d\theta$$

$$= \frac{1}{3} \int \frac{1}{\sec \theta \sqrt{9(\sec^2 \theta - 1)}} \tan \theta d\theta$$

$$= \frac{1}{3} \int \frac{1}{\sec \theta \sqrt{9 \tan^2 \theta}} \tan \theta d\theta$$

$$= \frac{1}{3} \int \frac{1}{\sec \theta \cdot 3 \tan \theta} \tan \theta d\theta$$

$$\frac{1}{9} \int \cos \theta d\theta$$

$$\frac{1}{9} \sin \theta + C$$

$$\frac{1}{9} \cdot \frac{\sqrt{x^2 - 9}}{x} + C$$

Question 4. (5 marks) Evaluate the indefinite integral:

$$\int \cot^3(2x) \csc^3(2x) dx = \int \cot^2(2x) \csc^2(2x) \cot(2x) \csc(2x) dx$$

$$= \int (\csc^2(2x) - 1) \csc^2(2x) \cot(2x) \csc(2x) dx$$

$$u = \csc(2x)$$

$$du = -\csc(2x) \cot(2x) (2) dx$$

$$-\frac{du}{2} = +\csc(2x) \cot(2x) dx$$

$$= \int (u^2 - 1) u^2 \left(-\frac{du}{2}\right)$$

$$= \frac{1}{2} \int -u^4 + u^2 du$$

$$= \frac{1}{2} \left[ -\frac{u^5}{5} + \frac{u^3}{3} \right] + C$$

$$= -\frac{\csc^5(2x)}{10} + \frac{\csc^3(2x)}{6} + C$$

Question 5. (5 marks) Evaluate the definite integral:

$$\int_{\pi/4}^{\pi/6} \cos^2 \theta d\theta = \int_{\pi/4}^{\pi/6} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{1}{2} \int_{\pi/4}^{\pi/6} 1 + \cos 2\theta d\theta$$

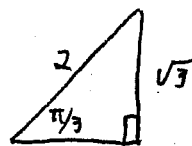
$$= \frac{1}{2} \left[ \theta + \frac{\sin 2\theta}{2} \right]_{\pi/4}^{\pi/6}$$

$$= \frac{1}{2} \left[ \frac{\pi}{6} + \frac{\sin \pi/3}{2} \right] - \frac{1}{2} \left[ \frac{\pi}{4} + \frac{\sin \pi/2}{2} \right]$$

$$= \frac{1}{2} \left[ \frac{\pi}{6} + \frac{\sqrt{3}/2}{2} \right] - \frac{1}{2} \left[ \frac{\pi}{4} + \frac{1}{2} \right]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8} - \frac{\pi}{8} - \frac{1}{4}$$

$$= \frac{\sqrt{3}-2}{8} - \frac{\pi}{24}$$



Question 6. (5 marks) Evaluate the limit:

$$\lim_{x \rightarrow 0^+} (\arctan 2x)^x$$

$$\text{Let } y = \lim_{x \rightarrow 0^+} (\arctan 2x)^x \quad \text{l.f. } 0^0$$

$$\ln y = \ln \lim_{x \rightarrow 0^+} (\arctan 2x)^x$$

$$\ln y = \lim_{x \rightarrow 0^+} \ln (\arctan 2x)^x$$

$$\ln y = \lim_{x \rightarrow 0^+} x \ln (\arctan 2x) \quad \text{l.f. } 0 \cdot -\infty$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\ln (\arctan 2x)}{1/x} \quad \text{l.f. } \frac{-\infty}{\infty}$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{1}{\arctan 2x} \cdot \frac{2}{1+4x^2} \quad \text{by } \hat{H}$$

$-\frac{1}{x^2}$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{-2x^2}{\arctan 2x (1+4x^2)} \quad \text{l.f. } \frac{0}{0}$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{-4x}{\frac{1}{1+4x^2} \cdot 2(1+4x^2) + \arctan(2x)(8x)} \quad \text{by } \hat{H}$$

$$\ln y = \frac{-4(0)}{2+0}$$

$$\ln y = 0$$

$$y = e^0$$

$$y = 1$$

$$\therefore \lim_{x \rightarrow 0^+} (\arctan 2x)^x = 1$$

Question 7. (5 marks) Determine whether the integral is convergent or divergent. Evaluate if convergent.

$$\int_0^{\infty} x e^{-2x} dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-2x} dx \quad \begin{array}{l} u = x \\ v = \frac{e^{-2x}}{-2} \end{array} \quad \begin{array}{l} du = dx \\ dv = e^{-2x} dx \end{array}$$

$$= \lim_{b \rightarrow \infty} \left[ \left[ \frac{x e^{-2x}}{-2} \right]_0^b - \int_0^b \frac{e^{-2x}}{-2} dx \right]$$

$$= \lim_{b \rightarrow \infty} \left[ \left[ \frac{b e^{-2b}}{-2} + \frac{0 e^{-2 \cdot 0}}{2} \right] + \frac{1}{2} \left[ \frac{e^{-2x}}{-2} \right]_0^b \right]$$

$$= \lim_{b \rightarrow \infty} \left[ \underbrace{\frac{-1}{2} b e^{-2b}}_{\text{l.f. } \infty \cdot 0} - \frac{1}{4} e^{-2b} + \frac{1}{4} \right]$$

$$= \frac{1}{4} + \lim_{b \rightarrow \infty} \frac{-1}{2} \frac{b}{e^{2b}} \quad \text{l.f. } \frac{\infty}{\infty}$$

$$= \frac{1}{4} + \lim_{b \rightarrow \infty} \left( \frac{-1}{2} \right) \frac{1}{e^{2b}} \quad \text{by } \hat{H}$$

$$= \frac{1}{4}$$

**Question 8.** (5 marks) Sketch the region enclosed by the given curves. Then find the area of the region.

$$y = \frac{3}{x}, y = 2x - 1, x = 1, x = e.$$

Intersection of  $y = \frac{3}{x}$ ,  $y = 2x - 1$  :

$$2x - 1 = \frac{3}{x}$$

$$(2x - 1)x = 3$$

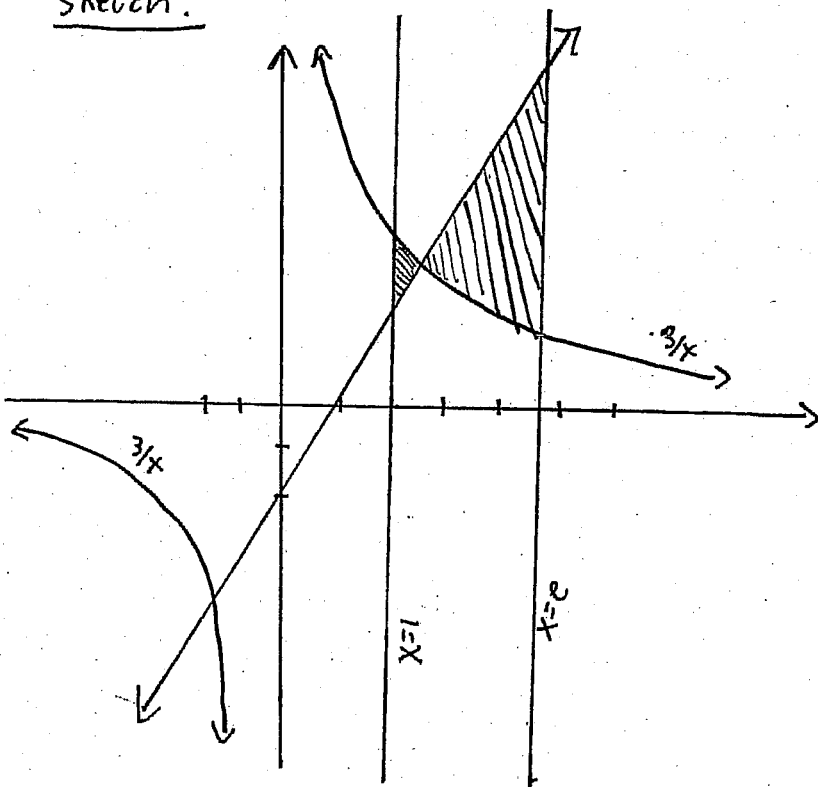
$$2x^2 - x = 3$$

$$2x^2 - x - 3 = 0$$

$$(2x - 3)(x + 1) = 0$$

$$x = \frac{3}{2} \quad x = -1$$

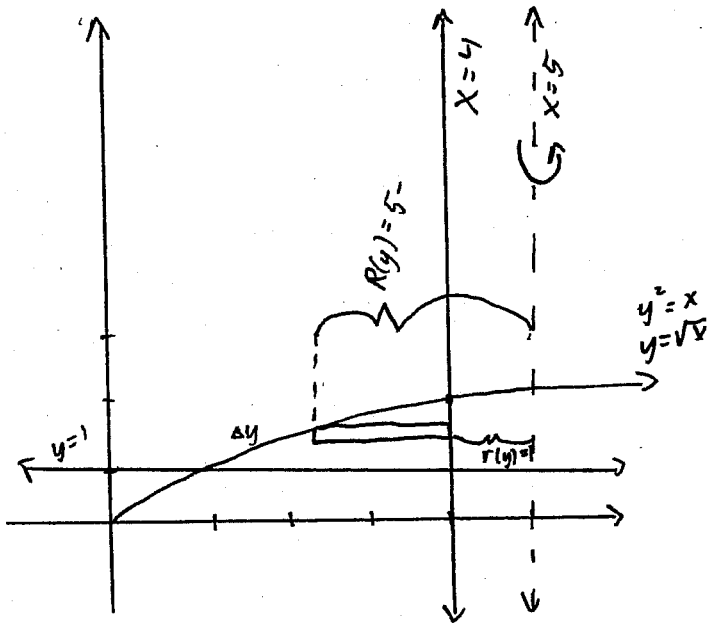
Sketch:



$$\begin{aligned} \text{Area} &= \int_1^{3/2} \frac{3}{x} - (2x - 1) dx \\ &+ \int_{3/2}^e (2x - 1) - \frac{3}{x} dx \\ &= \left[ 3 \ln|x| - x^2 + x \right]_1^{3/2} \\ &+ \left[ x^2 - x - 3 \ln|x| \right]_{3/2}^e \\ &= 3 \ln \frac{3}{2} - \left(\frac{3}{2}\right)^2 + \frac{3}{2} - 3 \ln 1 - 1^2 + 1 \\ &+ e^2 - e - 3 \ln e - \left(\frac{3}{2}\right)^2 + \frac{3}{2} + 3 \ln\left(\frac{3}{2}\right) \\ &= 6 \ln \frac{3}{2} - \frac{9}{2} + 3 + e^2 - e - 3 \\ &= 6 \ln \frac{3}{2} + e^2 - e - \frac{9}{2} \end{aligned}$$

**Question 9.** (5 marks) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y = \sqrt{x}$ ,  $y = 1$ ,  $x = 4$  about the  $x=5$



rep. element:

$$\begin{aligned} \Delta V &= \pi [(R(y))^2 - (r(y))^2] \Delta y \\ &= \pi [(5-y)^2 - 1^2] \Delta y \\ &= \pi [25 - 10y^2 + y^4 - 1] \Delta y \\ &= \pi [y^4 - 10y^2 + 24] \Delta y \end{aligned}$$

$$V = \int_1^2 \pi [y^4 - 10y^2 + 24] dy$$

$$= \pi \left[ \frac{y^5}{5} - \frac{10y^3}{3} + 24y \right]_1^2$$

$$= \pi \left[ \frac{2^5}{5} - \frac{10 \cdot 2^3}{3} + 24(2) \right]$$

$$- \pi \left[ \frac{1}{5} - \frac{10}{3} + 24 \right]$$

$$= \frac{103}{15} \pi$$



**Bonus Question.**

Given the following definite integral:

$$\int_1^2 x \ln x \, dx$$

a. (3 marks) For

$$\int_a^b f(x) \, dx$$

the error involved in using Simpson's Rule is

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where  $|f^{(4)}(x)| \leq K, \forall x \in [a, b]$ . How large do we have to choose  $n$  so that the approximation using Simpson's Rule is accurate to within 0.0001?b. (2 marks) Using the  $n$  determined in part a. use Simpson's Rule to approximate the integral.

c. (2 marks) Compare the result obtained in part b. to the actual value. Discuss any conclusion that arise from part a. and b.

a) Lets determine  $K$ .

$$f(x) = x \ln x$$

$$f'(x) = \ln x + x \frac{1}{x} = 1 + \ln x$$

$$f''(x) = \frac{1}{x}$$

$$f'''(x) = -\frac{1}{x^2}$$

$$f^{(4)}(x) = \frac{2}{x^3} \quad \therefore |f^{(4)}(x)| = \frac{2}{x^3} \leq K = 2 \quad \text{since } f^{(4)}(x) \text{ largest at } x=1 \text{ in } [1, 2]$$

So

$$\frac{K(b-a)^5}{180n^4} \leq 0.0001$$

$$\frac{2(2-1)^5}{180n^4} \leq$$

$$\frac{2(1)^5}{180(0.0001)} \leq n^4$$

$$3.24 \leq n$$

$$\therefore n = 4$$

b)

$$\int_1^2 x \ln x \, dx \approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_n)]$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \quad x_i = a + i\Delta x = 1 + i\left(\frac{1}{4}\right)$$

$$x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{6}{4}, x_3 = \frac{7}{4}, x_n = 2$$

$$= \frac{1}{12} [f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 4f\left(\frac{7}{4}\right) + f(2)]$$

$$= \frac{1}{12} [1 \ln 1 + 4\left(\frac{5}{4}\right) \ln\left(\frac{5}{4}\right) + 2\left(\frac{6}{4}\right) \ln\left(\frac{6}{4}\right) + 4\left(\frac{7}{4}\right) \ln\left(\frac{7}{4}\right) + 2 \ln 2]$$

$$\approx 0.636309$$

**Bonus Question.**

Given the following definite integral:

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b. (2 marks) Using the  $n$  determined in part a. use Simpson's Rule to approximate the integral.

c. (2 marks) Compare the result obtained in part b. to the actual value. Discuss any conclusion that arise from part a. and b.

$$\begin{aligned} c) \int_1^2 x \ln x \, dx &= \left[ uv \right]_1^2 - \int_1^2 v \, du & u &= \ln x & du &= \frac{1}{x} dx \\ & & v &= \frac{x^2}{2} & dv &= x \, dx \\ &= \left[ \frac{x^2}{2} \ln x \right]_1^2 - \int_1^2 \frac{x^2}{2} \cdot \frac{1}{x} \, dx \\ &= \frac{2^2}{2} \ln 2 - \frac{1^2 \ln 1}{2} - \left[ \frac{x^2}{4} \right]_1^2 \\ &= 2 \ln 2 - \frac{4}{4} + \frac{1}{4} \\ &= \ln 4 - 1 + \frac{1}{4} \\ &\doteq 0.636294 \end{aligned}$$

$$\begin{aligned} \text{Error} &= |\text{actual} - \text{estimated}| \\ &= |0.636294 - 0.636309| \\ &= 0.000014 \end{aligned}$$

∴ Like expected because of the choice of  $n$  the approx is accurate to within 0.0001.