Dawson College: Calculus II: 201-NYB-05-S03: Fall 2010

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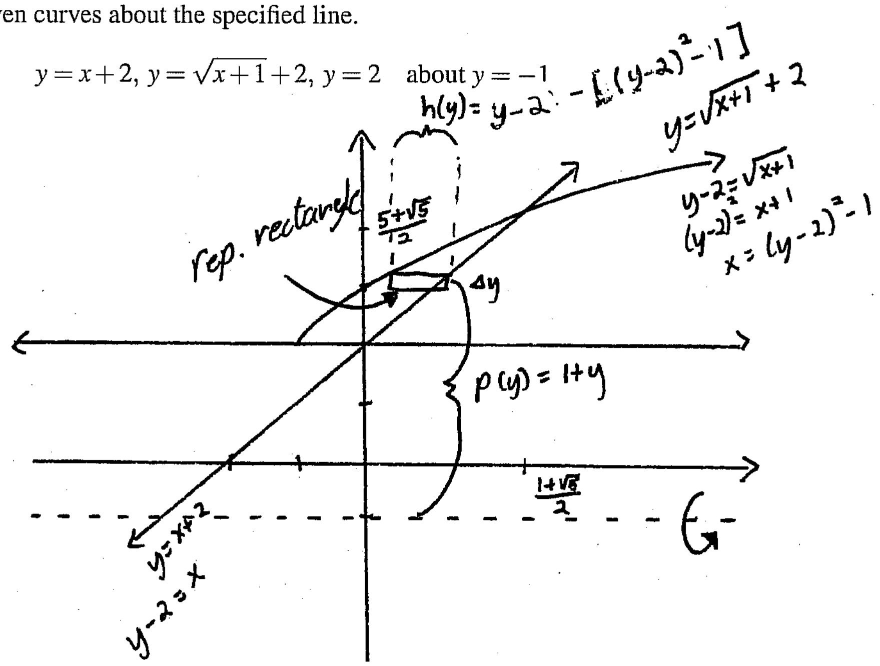
Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y = x+2$, $y = \sqrt{x+1}+2$, $y = 2$ about $y = -1$



Intersection:

$x+2 = \sqrt{x+1} + 2$

$x = \sqrt{x+1}$

$x^2 = (\sqrt{x+1})^2$

$0 = x^2 - x - 1$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

rep. element: $\Delta V = 2\pi p(y) h(y) \Delta y$

$= 2\pi (1+y)\left[y-2 - [(y-2)^2-1]\right]\Delta y$

$$V = \int_2^{\frac{5+\sqrt{5}}{2}} 2\pi (1+y)\left[y-2 - [(y-2)^2-1]\right] dy$$

Question 2. *(5 marks)* Find the length of the curve.

$$y = \frac{x^5}{6} + \frac{1}{10x^3}, \quad 1 \le x \le 2 \qquad y' = \frac{5x^4}{6} - \frac{3}{10x^4}$$

$$S = \int_1^2 \sqrt{1 + (y')^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{25x^8}{36} - \frac{1}{2} + \frac{9}{100x^8}}\, dx$$

$$= \int_1^2 \sqrt{\frac{25x^8}{36} + \frac{1}{2} + \frac{9}{100x^8}}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{5}{6}x^4 + \frac{3}{10x^4}\right)^2}\, dx$$

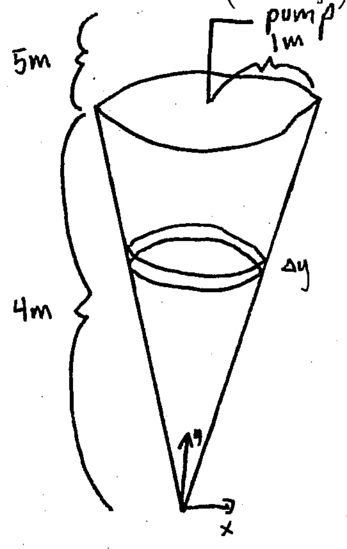
$$= \int_1^2 \frac{5}{6}x^4 + \frac{3}{10x^4}\, dx$$

$$= \left[\frac{x^5}{6} - \frac{1}{10x^3}\right]_1^2$$

$$= \left[\frac{2^5}{6} - \frac{1}{10\,2^3}\right] - \left[\frac{1}{6} - \frac{1}{10}\right]$$

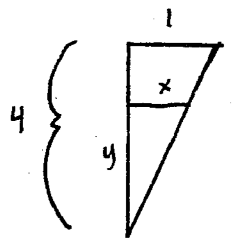
$$= \frac{1261}{240}$$

Question 3. *(5 marks)* A conic tank is filled with liquid which has a density of $\rho = 1000\frac{kg}{m^3}$. If the tank is 2$m$ across the top and has a height of 4m, set up the integral that represent the work performed to empty the tank of through a pipe that extends 5m above the top edge? $\left(g = 9.8\frac{m}{s^2}\right)$



volume of slice:

$$\Delta V = \pi r^2 \Delta y$$
$$= \pi x^2 \Delta y$$
$$= \pi \left(\frac{y}{4}\right)^2 \Delta y$$
$$= \frac{\pi}{16} y^2 \Delta y$$



$$\frac{x}{1} = \frac{y}{4}$$

mass of slice:

$$\Delta m = \Delta V \rho$$
$$= 1000 \frac{\pi}{16} y^2 \Delta y$$

force of slice:

$$\Delta F = \Delta m g$$
$$= \frac{1000 \pi g y^2}{16} \Delta y$$
$$= \frac{9800\pi}{16} y^2 \Delta y$$

distance from slice to pump: $d = 9 - y$

Work:

$$\Delta W = \Delta F d$$
$$= \frac{9800\pi}{16} y^2 (9-y) \Delta y$$

$$W = \int_0^4 \frac{9800\pi}{16} y^2 (9-y)\, dy$$

Question 4. *(2 marks)* Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{1, -2, \frac{3}{2}, -\frac{4}{6}, \frac{5}{24}, \ldots\right\}$$

$$a_n = (-1)^{n+1} \frac{n}{(n-1)!}$$

Question 5. Suppose that you and Yann have infinite life. If Yann gives you 25 grams of Krypton (not to be confused with Kryptonite) on the first day, then 3/4 of that amount on the second day, then on the third day, you are given 3/4 of the amount of the second day, and if this process continues forever.

a. *(1 mark)* Will Yann give you an infinite amount of Krypton?

b. *(2 marks)* What is the amount of Krypton given to you by the process described above?

c. *(bonus 1 mark)* Does Yann have an infinite amount of Krypton, Justify.

a) $25 + 25\left(\frac{3}{4}\right) + 25\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) + 25\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) + \ldots$

$= 25 + 25\left(\frac{3}{4}\right) + 25\left(\frac{3}{4}\right)^2 + 25\left(\frac{3}{4}\right)^3 + \ldots$

$= \sum_{n=0}^{\infty} 25\left(\frac{3}{4}\right)^n \leftarrow$ geometric series

Since $r = \frac{3}{4}$ and $0 < |r| < 1$ the series converges

$\therefore$ Yann gives you a finite amount of Krypton.

b) $\sum_{n=0}^{\infty} 25\left(\frac{3}{4}\right)^n = \frac{a}{1-r}$ $\qquad a = 25$, $r = \frac{3}{4}$

$= \frac{25}{1-\frac{3}{4}}$

$= \frac{25}{\frac{1}{4}}$

$= 100$

$\therefore$ Yann gives you 100 grams of Krypton

c) It is inconclusive based on a), b)

Question 6. *(5 marks)* Determine whether each of the following series converges or diverges. Justify your answer.

a. *(5 marks)*

$$\sum_{n=10}^{\infty} \underbrace{\frac{\sqrt[3]{n^9+3n^4+4}}{n\sqrt{n^9+n^4}}}_{a_n}$$

Test series: $\sum_{n=10}^{\infty} b_n$ where $b_n = \frac{\sqrt[3]{n^9}}{n\sqrt{n^9}} = \frac{n^{9/3}}{n\,n^{9/2}} = \frac{n^{\cancel{3}2}}{\cancel{n}n^{9/2}} = \frac{1}{n^{5/2}}$

p-series where $p = \frac{5}{2} > 1$ $\therefore$ converges

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt[3]{n^9+3n^4+4}}{n\sqrt{n^9+n^4}}}{\frac{\sqrt[3]{n^9}}{n\sqrt{n^9}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^9+3n^4+4}}{\cancel{n}\sqrt{n^9+n^4}} \cdot \frac{\cancel{n}\sqrt{n^9}}{\sqrt[3]{n^9}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^9+3n^4+4}}{\sqrt[3]{n^9}} \cdot \frac{\sqrt{n^9}}{\sqrt{n^9+n^4}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^9+3n^4+4}}{\sqrt[3]{n^9}} \cdot \lim_{n\to\infty} \frac{\sqrt{n^9}}{\sqrt{n^9+n^4}}$$

$= 1 \cdot 1 = 1 > 0$ and finite

$\therefore \sum_{n=10}^{\infty} a_n$ converges by Limit comparison test.

b. *(5 marks)*

$$\sum_{n=1}^{\infty} \frac{e^{-n^{1/4}}}{n^{3/4}}$$

Let $f(x) = \frac{e^{-x^{1/4}}}{x^{3/4}}$

- $f(x)$ continuous on $[1,\infty)$? yes
- $f(x)$ positive on $[1,\infty)$? yes
- $f(x)$ decreasing on $[1,\infty)$? yes

$$f'(x) = \frac{e^{-x^{1/4}}\frac{1}{4}(-x^{-3/4}) - e^{-x^{1/4}}\frac{3}{4}x^{-1/4}}{(x^{3/4})^2} < 0$$

$$\int_1^{\infty} \frac{e^{-x^{1/4}}}{x^{3/4}}\,dx$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{e^{-x^{1/4}}}{x^{3/4}}\,dx$$

$u = -x^{1/4}$

$du = \frac{-1}{4x^{3/4}}\,dx$

$u(1) = -1^{1/4} = -1$

$u(b) = -b^{1/4}$

$$= \lim_{b\to\infty} \int_{-1}^{-b^{1/4}} e^u(-4)\,du$$

$$= \lim_{b\to\infty} -4\int_{-1}^{-b^{1/4}} e^u\,du$$

$$= -4\lim_{b\to\infty} \left[e^u\right]_{-1}^{-b^{1/4}}$$

$$= -4\lim_{b\to\infty} \left[\cancelto{0}{e^{-b^{1/4}}} - e^{-1}\right]$$

$= \frac{4}{e}$ ← converges

$\therefore \sum_{n=1}^{\infty} \frac{e^{-1/4}}{n^{3/4}}$ converges by integral test.

Question 7. *(5 marks)* Find the sum of the following series if it converges or show it diverges.

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$$

Let $a_n = \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$$

$$= \sin\left(\frac{\pi}{2}\right)$$

$$= 1$$

$$\neq 0$$

$\therefore$ does not converge by n^{th} term divergence test.

Question 8. *(5 marks)* Find the Taylor Polynomial of order 2 of $f(x) = \arctan(3x)$ at $x = \frac{1}{3}$.

$$P_3(x) = f(c) + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!}$$

$f(x) = \arctan(3x)$

$f\left(\frac{1}{3}\right) = \arctan\left(3\left(\frac{1}{3}\right)\right) = \frac{\pi}{4}$

$f'(x) = \frac{1}{1+(3x)^2} \cdot 3$

$f'\left(\frac{1}{3}\right) = \frac{1}{1+\left(3\frac{1}{3}\right)^2} = \frac{3}{2}$

$f''(x) = \frac{-3}{\left(1+(3x)^2\right)^2} 2(3x)\cdot 3 = \frac{-54x}{\left(1+(3x)^2\right)^2}$

$f''\left(\frac{1}{3}\right) = \frac{-54\left(\frac{1}{3}\right)}{\left(1+\left(3\frac{1}{3}\right)^2\right)^2} = \frac{-18}{2^2} = \frac{-9}{2}$

$$P_3(x) = \frac{\pi}{4} + \frac{3}{2}\left(x-\frac{1}{3}\right) + \frac{-9/2}{2}\left(x-\frac{1}{3}\right)^2$$

$$= \frac{\pi}{4} + \frac{3}{2}\left(x-\frac{1}{3}\right) - \frac{9}{4}\left(x-\frac{1}{3}\right)^2$$

Bonus Question. *(3 marks)*
Prove: If

$$\sum_{n=k}^{\infty} a_n$$

converges then

$$\lim_{n\to\infty} a_n = 0.$$

If $\sum_{n=k}^{\infty} a_n$ converges then $S = \sum_{n=K}^{\infty} a_n$

and $S = \lim_{n\to\infty} S_n$ (1)

where $S_n = a_K + a_{K+1} + a_{K+2} + \dots + a_{n-1} + a_n$

so $S_{n-1} = a_K + a_{K+1} + a_{K+2} + \dots + a_{n-1}$ and it follows that

$$S = \lim_{n\to\infty} S_{n-1} \quad (2)$$

So (1) - (2)

$$S - S = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$0 = \lim_{n\to\infty} a_n$$

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