Dawson College: Calculus II: 201-NYB-05-S03: Fall 2010

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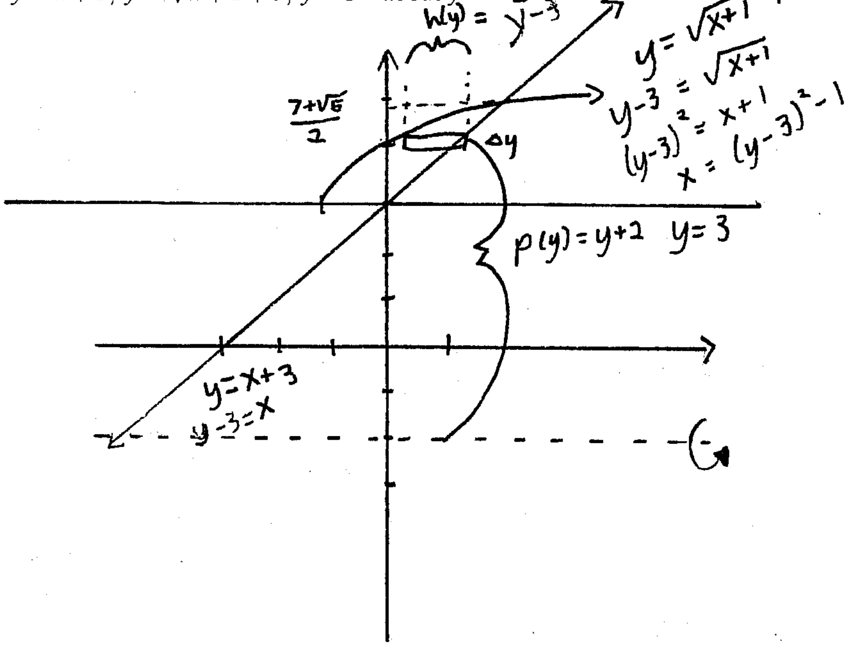
Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y = x+3$, $y = \sqrt{x+1}+3$, $y = 3$ about $y = -2$

$h(y) = y-3-\left[(y-3)^2-1\right]$



<u>Intersection:</u>

$$x+3 = \sqrt{x+1}+3$$
$$x = \sqrt{x+1}$$
$$x^2 = x+1$$
$$0 = x^2-x-1$$
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2-4(1)(-1)}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

<u>rep. element.</u>

$$\Delta V = 2\pi p(y) h(y) \Delta y$$
$$= 2\pi (y+2)\left[y-3-\left[(y-3)^2-1\right]\right]\Delta y$$

$$V = \int_3^{\frac{7+\sqrt{5}}{2}} 2\pi (y+2)\left[y-3-\left[(y-3)^2-1\right]\right]dy$$

Question 2. *(5 marks)* Find the length of the curve.

$$y = \ln(\sec x),\ 0 \le x \le \frac{\pi}{4}$$

$$y' = \frac{1}{\sec x} \sec x \tan x = \tan x$$

$$S = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec x\, dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= \ln|\sec \tfrac{\pi}{4} + \tan\tfrac{\pi}{4}| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

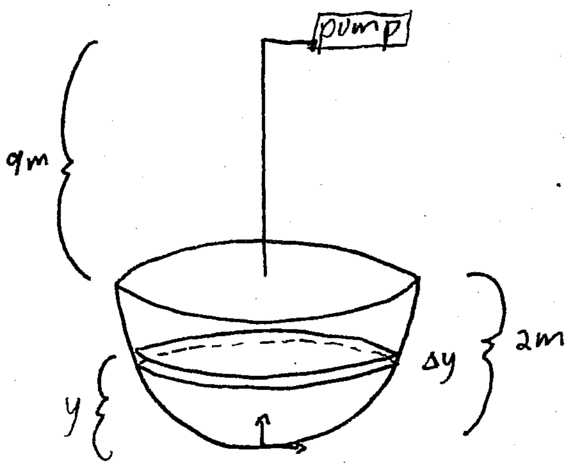
$$= \ln(\sqrt{2} + 1)$$

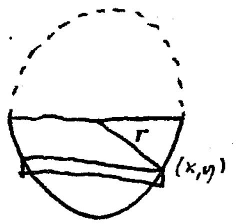
Question 3. *(5 marks)* A hemispherical tank is filled with a liquid which has a density of $\rho = 1000\frac{kg}{m^3}$. If the tank is $4m$ across the top *(diameter)*, set up the integral that represent the work performed to empty the tank through a pipe that extends $9m$ above the top edge? $\left(g = 9.8\frac{m}{s^2}\right)$

volume of slice: $\Delta V = \pi r^2 \Delta y$

$= \pi x^2 \Delta y$

$= \pi \left[2^2 - (y-2)^2\right] \Delta y$





$x^2 + (y-2)^2 = 2^2$

$x^2 = 2^2 - (y-2)^2$

mass of slice: $\Delta m = \rho \Delta V$

$= 1000\pi \left[2^2 - (y-2)^2\right] \Delta y$

force of slice: $\Delta F = \Delta m g$

$= 9800\pi \left[2^2 - (y-2)^2\right] \Delta y$

distance of slice to pump: $d = 11 - y$

Work:

$\Delta W = \Delta F d$

$= 9800\pi \left[2^2 - (y-2)^2\right](11-y)\Delta y$

$$W = \int_0^2 9800\pi \left[2^2 - (y-2)^2\right](11-y)\,dy$$

Question 4. *(2 marks)* Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{1, -\frac{1}{2}, \frac{2}{3}, -\frac{6}{4}, \frac{24}{5}, \ldots\right\}$$

$$a_n = (-1)^{n+1} \frac{(n-1)!}{n}$$

Question 5. Suppose that you and Yann have infinite life. If Yann gives you 20 grams of Krypton (not to be confused with Kryptonite) on the first day, then $3/5$ of that amount on the second day, then on the third day, you are given $3/5$ of the amount of the second day, and this process continues forever.

a. *(1 mark)* Will Yann give you an infinite amount of Krypton?

b. *(2 marks)* What is the amount of Krypton given to you by the process described above?

c. *(bonus 1 mark)* Does Yann have an infinite amount of Krypton, Justify.

a) $20 + 20\left(\frac{3}{5}\right) + 20\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) + 20\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) + \ldots$

$= 20 + 20\left(\frac{3}{5}\right) + 20\left(\frac{3}{5}\right)^2 + 20\left(\frac{3}{5}\right)^3 + \ldots$

$= \sum_{n=0}^{\infty} 20\left(\frac{3}{5}\right)^n$

The above is a geometric series where $r = \frac{3}{5}$. and since $0 < |r| < 1$ the series converges. Hence Yann gives you a finite amount of Krypton.

b) $\sum_{n=0}^{\infty} 20\left(\frac{3}{5}\right)^n = \frac{a}{1-r}$

$= \frac{20}{1-\frac{3}{5}} = \frac{20}{\frac{2}{5}} = 50$

c) It is inconclusive based on a), b).

Question 6. *(5 marks)* Determine whether each of the following series converges or diverges. Justify your answer.

a. *(5 marks)*

$$\sum_{n=10}^{\infty} \underbrace{\frac{n\sqrt[3]{n^9+3n^4+4}}{\sqrt{n^9+n^4}}}_{a_n}$$

Test series: $\sum b_n$ where $b_n = \frac{n\sqrt[3]{n^9}}{\sqrt{n^9}} = \frac{n^4}{\sqrt{n^9}} = \frac{1}{n^{1/2}}$

is a p-series where $p = \frac{1}{2} \leq 1$ $\therefore$ diverges

$$\lim_{n\to\infty} \frac{\frac{n\sqrt[3]{n^9+3n^4+4}}{\sqrt{n^9+n^4}}}{\frac{n\sqrt[3]{n^9}}{\sqrt{n^9}}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\sqrt[3]{n^9+3n^4+4}}{\sqrt{n^9+n^4}} \cdot \frac{\sqrt{n^9}}{\cancel{n}\sqrt[3]{n^9}}$$

$$\lim_{n\to\infty} \sqrt[3]{\frac{n^9+3n^4+4}{n^9}} \cdot \sqrt{\frac{n^9}{n^9+n^4}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n^9+3n^4+4}{n^9}} \cdot \lim_{n\to\infty} \sqrt{\frac{n^9}{n^9+n^4}}$$

$= 1 \cdot 1 > 0$ and finite

$\therefore$ limit comparison test the series diverges.

b. *(5 marks)*

$$\sum_{n=1}^{\infty} \frac{e^{-n^{1/5}}}{n^{4/5}}$$

Let $f(x) = \frac{e^{-x^{1/5}}}{x^{4/5}}$

- $f(x)$ positive on $[1,\infty)$? yes
- $f(x)$ continuous on $[1,\infty)$? yes
- $f(x)$ decreasing on $[1,\infty)$? yes

$$f'(x) = \frac{e^{-x^{1/5}} \cdot -\frac{1}{5}x^{-4/5} - \frac{4}{5}x^{-1/5}e^{-x^{1/5}}}{(x^{4/5})^2}$$

$$= \frac{e^{-x^{1/5}}\left[-\frac{1}{5}x^{-4/5} - \frac{4}{5}x^{-1/5}\right]}{x^{8/5}} < 0$$

$$\int_1^{\infty} \frac{e^{-x^{1/5}}}{x^{4/5}}\,dx = \lim_{b\to\infty}\int_1^b \frac{e^{-x^{1/5}}}{x^{4/5}}\,dx$$

$u = -x^{1/5}$ $\quad u(1) = -1^{1/5}$

$du = \frac{-1}{5x^{4/5}}dx$ $\quad = -1$

$u(b) = -b^{1/5}$

$-5\,du = \frac{1}{x^{4/5}}dx$

$$= \lim_{b\to\infty}\int_{-1}^{-b^{1/5}} e^u(-5)\,du$$

$$= -5\lim_{b\to\infty}\left[e^u\right]_{-1}^{-b^{1/5}}$$

$$= -5\lim_{b\to\infty}\left[\cancel{e^{-b^{1/5}}} - e^{-1}\right]$$

$= \frac{5}{e}$ ← converged

$\therefore$ by integral test the series converges

Question 7. *(5 marks)* Find the sum of the following series if it converges or show it diverges.

$$\sum_{n=1}^{\infty} \cos\left(2\pi - \frac{1}{n}\right)$$

Let $a_n = \cos\left(2\pi - \frac{1}{n}\right)$

$\lim_{n \to \infty} a_n$

$= \lim_{n \to \infty} \cos\left(2\pi - \frac{1}{n}\right)$

$= \cos(2\pi)$

$= 1$

$\neq 0$

$\therefore$ diverges by the n^{th} term divergence test.

Question 8. *(5 marks)* Find the Taylor Polynomial of order 2 of $f(x) = \arctan(5x)$ at $x = \frac{1}{5}$.

$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!}$

$f(x) = \arctan(5x)$

$f(\frac{1}{5}) = \arctan(5(\frac{1}{5})) = \arctan 1 = \frac{\pi}{4}$

$f'(x) = \frac{5}{1+(5x)^2}$

$f'(\frac{1}{5}) = \frac{5}{1+(5\frac{1}{5})^2} = \frac{5}{2}$

$f''(x) = \frac{-5 \quad 2(5x)\cdot 5}{(1+(5x)^2)^2}$

$= \frac{-250x}{(1+(5x)^2)^2}$

$f''(\frac{1}{5}) = \frac{-250(\frac{1}{5})}{(1+(5(\frac{1}{5}))^2)^2} = \frac{-50}{4} = \frac{-25}{2}$

$P_2(x) = \frac{\pi}{4} + \frac{5}{2}(x - \frac{1}{5}) + \frac{-25}{4}(x - \frac{1}{5})^2$

Bonus Question. *(3 marks)*
Prove: If

$$\sum_{n=k}^{\infty} a_n$$

onverges then

$$\lim_{n\to\infty} a_n = 0.$$

If $\sum_{n=k}^{\infty} a_n$ converges then $S = \sum_{n=K}^{\infty} a_n$

and $S = \lim_{n\to\infty} S_n$ ①

where $S_n = a_K + a_{K+1} + a_{K+2} + \dots + a_{n-1} + a_n$

so $S_{n-1} = a_K + a_{K+1} + a_{K+2} + \dots + a_{n-1}$ and it follows that

$$S = \lim_{n\to\infty} S_{n-1} \quad ②$$

So ① - ②

$$S - S = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$0 = \lim_{n\to\infty} a_n$$

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