1 a) $\lim_{n\to\infty} \left(3 + \underset{\downarrow 0}{\frac{2}{n}}\right)^{n \to n} = \infty$

b)

LET $y = \left(1+\frac{2}{n}\right)^n \Rightarrow \ln y = n \ln\left(1+\frac{2}{n}\right)$

$\therefore \lim_{n\to\infty} \ln y = \lim_{n\to\infty} n \ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{1/n}$ I.F. $\frac{0}{0}$

(H) $\lim_{n\to\infty} \frac{\frac{1}{1+2/n}\cdot\left(-2/n^2\right)}{-1/n^2} = \lim_{n\to\infty} \frac{2}{1+2/n} = \frac{2}{1+0} = 2$

$\therefore \lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = e^{\lim_{n\to\infty} \ln y} = e^2$

#2 a) $\sum_{n=2}^{\infty} \frac{3\cdot 6^n + 4\cdot 6^{n+1}}{9^{n-1}} = \sum_{n=2}^{\infty}\left(\frac{3\cdot 6^n}{9^{n-1}} + \frac{4\cdot 6^{n+1}}{9^{n-1}}\right) = \sum a_n$

NOW $\sum_{n=2}^{\infty} \frac{3\cdot 6^n}{9^{n-1}} = \sum_{n=2}^{\infty} \frac{3\cdot 6\cdot 6^{n-1}}{9^{n-1}} = \sum_{n=2}^{\infty} 18\cdot\left(\frac{6}{9}\right)^{n-1}$

$= \sum_{n=2}^{\infty} 18\cdot\left(\frac{2}{3}\right)^{n-1} = \sum_{n=2}^{\infty} 18\cdot\left(\frac{2}{3}\right)^{n-1} + 18 - 18$

$= \sum_{n=1}^{\infty} 18\left(\frac{2}{3}\right)^{n-1} - 18$ (GEOMETRIC, $r = 2/3$)

$= \frac{18}{1-2/3} - 18 = \frac{18}{1/3} - 18 = 36.$

AND $\sum_{n=2}^{\infty} \frac{4\cdot 6^{n+1}}{9^{n-1}} = \sum_{n=2}^{\infty} \frac{4\cdot 6^2\cdot 6^{n-1}}{9^{n-1}} = \sum_{n=2}^{\infty} 144\left(\frac{2}{3}\right)^{n-1}$

$= \sum_{n=2}^{\infty} 144\left(\frac{2}{3}\right)^{n-1} + 144 - 144$

$= \sum_{n=1}^{\infty} 144\left(\frac{2}{3}\right)^{n-1} - 144$ (GEOMETRIC, $r = 2/3$)

$= \frac{144}{1-2/3} - 144 = \frac{144}{1/3} - 144 = 288$

$\therefore \sum a_n = \sum_{n=2}^{\infty} \frac{3\cdot 6^n}{9^{n-1}} + \sum_{n=2}^{\infty} \frac{4\cdot 6^{n+1}}{9^{n-1}}$

$= 36 + 288 = 324$

$\therefore$ THE SERIES CONVERGES.

b) $S_n = [\arctan(2) - \arctan(1)] + [\arctan(3) - \arctan(2)]$

$+ [\arctan(4) - \arctan(3)] + [\arctan(5) - \arctan(4)]$

$+ \ldots + [\arctan(n-2) - \arctan(n-3)] + [\arctan(n-1) - \arctan(n-$

$+ [\arctan(n) - \arctan(n-1)] + [\arctan(n+1) - \arctan(n)]$

$= -\arctan(1) + \arctan(n+1)$

$\therefore \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan(n)] = \lim_{n \to \infty} S_n$

$= \lim_{n \to \infty} [-\arctan(1) + \arctan(n+1)]$

$= -\pi/4 + \pi/2 = \pi/4$

$\therefore$ THE SERIES CONVERGES.

#3 a)

$$a_K = \frac{(3K-2)(K^2-5)}{(K+1)(K^3+1)^2} < \frac{(3K)(K^2)}{K(K^3)^2} = \frac{3K^3}{K^7} = \frac{3}{K^4} = b_K$$

NOW $\sum_{K=1}^{\infty} b_K = \sum_{K=1}^{\infty} \frac{3}{K^4} = 3 \cdot \sum_{K=1}^{\infty} \frac{1}{K^4}$ CONVERGES

(P-SERIES, $p = 4 > 1$)

$\therefore \sum_{K=1}^{\infty} a_K$ CONVERGES BY COMPARISON TEST.

b) LET $a_n = \frac{2^{n+1}(\ln n)^n}{n^n} = \frac{2 \cdot 2^n (\ln n)^n}{n^n}$

$$\therefore \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{2\cdot 2^n (\ln n)^n}{n^n}\right|}$$

$$= \lim_{n\to\infty} \frac{2^{1/n} \cdot 2 \cdot \ln n}{n} = \lim_{n\to\infty} 2 \cdot 2^{1/n} \cdot \lim_{n\to\infty} \frac{\ln n}{n}$$

I.F. $\frac{\infty}{\infty}$

$$\overset{H}{=} \lim_{n\to\infty} 2 \cdot 2^{1/n} \cdot \lim_{n\to\infty} \frac{1/n}{1} = 2 \cdot 1 \cdot 0 = 0 < 1$$

$\therefore$ THE SERIES $\sum a_n$ CONVERGES (ABSOLUTELY).

c) LET $f(x) = \frac{\ln x}{x^4}$

$$f'(x) = \frac{\frac{1}{x}\cdot x^4 - \ln x \cdot 4x^3}{(x^4)^2} = \frac{x^3 - 4x^3 \ln x}{x^8}$$

$$= \frac{x^3(1-4\ln x)}{x^8} < 0 \iff 1 - 4\ln x < 0 \quad \text{(SINCE } x>$$

$$\iff 1 < 4\ln x \iff \tfrac{1}{4} < \ln x \iff x > e^{1/4}$$

SO f IS (EVENTUALLY DECREASING). IT IS ALSO POSITIVE ON $[1,\infty)$ AND CONTINUOUS ON $[1,\infty)$. SO WE CAN USE INTEGRAL TEST.

NOW $I = \int_1^\infty \frac{\ln x}{x^4}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^4}\,dx$

FIRST $\int \frac{\ln x}{x^4}\,dx$

LET
$u = \ln x \qquad dv = \frac{1}{x^4}\,dx$
$du = \frac{1}{x}\,dx \qquad v = \frac{x^{-3}}{-3}$

$$= -\frac{\ln x}{3x^3} - \int \left(\frac{-1}{3x^3}\right)\cdot\frac{1}{x}\,dx$$

$$= -\frac{\ln x}{3x^3} + \frac{1}{3}\int \frac{1}{x^4}\,dx = \frac{-\ln x}{3x^3} + \frac{1}{3}\left(\frac{x^{-3}}{-3}\right) + C$$

$$= \frac{-\ln x}{3x^3} - \frac{x^{-3}}{9} + C$$

$$\therefore I = \lim_{t\to\infty}\left[\frac{-\ln x}{3x^3} - \frac{1}{9x^3}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-\ln t}{3t^3} - \frac{1}{9t^3} + \frac{\ln 1}{3(1)^3} + \frac{1}{9(1)^3}\right]$$

NOW $\lim_{t\to\infty} \frac{\ln t}{3t^3} \overset{H}{=} \lim_{t\to\infty} \frac{1/t}{9t^2} = \lim_{t\to\infty} \frac{1}{9t^3} = 0$

I. F. $\frac{\infty}{\infty}$

$\therefore\ I = 0 - 0 + 0 + \frac{1}{9} = +1/9$

$\therefore$ THE SERIES CONVERGES BY INTEGRAL TEST

4 LET $a_n = (-1)^n \frac{n}{n^2+1}$

THEN $\sum |a_n| = \sum \left| (-1)^n \frac{n}{n^2+1} \right| = \sum \frac{n}{n^2+1}$

LET $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 > 0$$

SINCE $\sum b_n$ DIVERGES (p-SERIES, $p=1$)

SO DOES $\sum \frac{n^2}{n^2+1}$ BY LIMIT COMPARISON TEST.

$\therefore$ $\sum a_n$ IS NOT ABSOLUTELY CONVERGENT.

IS IT CONDITIONALLY CONVERGENT?

<u>ALTERNATING SERIES TEST</u>:

LET $c_n = \frac{n}{n^2+1}$

1) $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = \frac{0}{1+0} = 0$

1) LET $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{1-x^2}{(1+x^2)^2} \leq 0$

WHEN $x \geq 1$

$\therefore$ c_n IS DECREASING

$\therefore$ $\sum a_n$ IS CONVERGENT BY ALTERNATING SERIES TEST

$\therefore$ $\sum a_n$ IS CONDITIONALLY CONVERGENT.

#5 $a_n = \frac{n}{5^n}(x+2)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}(x+2)^{n+1} \cdot \frac{5^n}{n(x+2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{(x+2)}{5}\right| = \frac{|x+2|}{5} < 1$$

$\Rightarrow |x+2| < 5$ BY RATIO TEST THE RADIUS OF CONVERGENCE IS $R=5$

NOW $-5 < x+2 < 5$

$-7 < x < 3$

WHEN $x=-7$

$$\sum_{n=1}^{\infty}\frac{n}{5^n}(-7+2)^n = \sum_{n=1}^{\infty}\frac{n(-1)^n5^n}{5^n} = \sum_{n=1}^{\infty}(-1)^n\cdot n$$

WHICH DIVERGES SINCE $\lim_{n\to\infty}(-1)^n\cdot n \neq 0$ (TEST FOR DIVERGENCE

WHEN $x=3$

$$\sum_{n=1}^{\infty}\frac{n}{5^n}(3+2)^n = \sum_{n=1}^{\infty} n$$ WHICH DIVERGES SINCE

$\lim_{n\to\infty} n = \infty \neq 0$ (TEST FOR DIVERGENCE.

$\therefore$ THE ~~RADIUS~~ INTERVAL OF CONVERGENCE IS $(-7,3)$.

#6a) $\frac{1}{4+x^2} = \frac{1}{4(1+\frac{x^2}{4})} = \frac{1}{4} \cdot \frac{1}{1+(\frac{x}{2})^2}$

$= \frac{1}{4} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2)^{2n}}$

$= \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^{n+1}}$

SINCE $\frac{1}{1+x^2}$ CONVERGES IF AND ONLY IF $-1 < x < 1$

OUR NEW POWER SERIES CONVERGES IF AND ONLY IF

$-1 < \frac{x}{2} < 1 \iff -2 < x < 2$

SO IT HAS RADIUS OF CONVERGENCE $R = 2$ AND INTERVAL OF CONVERGENCE $(-2, 2)$

b) $\frac{x}{1+x^2} = x \cdot \frac{1}{1+x^2} = x \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$

NOW, $\frac{1-x^2}{(1+x^2)^2} = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n x^{2n+1}\right)$

$= \sum_{n=0}^{\infty} \frac{d}{dx}\left((-1)^n x^{2n+1}\right)$

$= \sum_{n=0}^{\infty} (-1)^n (2n+1) \cdot x^{2n}$

THIS POWER SERIES HAS THE SAME RADIUS OF CONVERGENCE ~~AS~~ $\frac{x}{1+x^2}$ AND SO THE SAME AS $\frac{1}{1+x^2}$ WHICH IS $R=1$.

WE CANNOT SAY IF THE END POINTS $x = 1, -1$ ARE INCLUDED IN THE INTERVAL SO WE DON'T KNOW THE

INTERVAL OF CONVERGENCE.