Dawson College: Calculus III: 201-BZF-05 S01

October 28, 2011

Test 2

Question 1. *(5 marks)* Find the Maclauren Series of the following function using the definition:

$$f(x) = \frac{1}{(1+2x)^2} = (1+2x)^{-2}$$

$f(0) = 1$

$f'(x) = -2(1+2x)^{-3}(2)$

$f'(0) = -2 \cdot 2$

$f''(x) = (-2)(-3)(1+2x)^{-4} \cdot (-2)^2$

$f''(0) = 2 \cdot 3 \cdot 2^2$

$f'''(x) = (-2)(-3)(-4)(1+2x)^{-5}(2)^3$

$f'''(0) = -2 \cdot 3 \cdot 4 \cdot 2^3$

$f^{(4)}(x) = (-2)(-3)(-4)(-5)(1+2x)^{-6}(2)^4$

$f^{(4)}(0) = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 2^4$

$\vdots$

$f^{(n)}(x) = (-1)^n (n+1)! (1+2x)^{-n-2} \cdot (2)^n$

$f^{(n)}(0) = (-1)^n (n+1)! \cdot 2^n$

$\therefore$ THE MACLAUREN SERIES FOR $f(x)$ IS

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)! 2^n}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) \cdot 2^n \cdot x^n$$

Question 2. *(6 marks)* Use series to approximate the following definite integral so that the |error| < 0.0001. Clearly justify the use of any theorems.

$f(x) = x^3 e^{-x^3}$, $\int_0^{0.5} f(x)dx$

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$$

$$\therefore x^3 e^{-x^3} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n!}$$

$$\therefore \int x^3 e^{-x^3} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+4}}{(3n+4)n!} + C$$

$$\therefore \int_0^{0.5} x^3 e^{-x^3} dx = \sum_{n=0}^{\infty} \frac{(-1)^n (0.5)^{3n+4}}{(3n+4)n!} - 0 = \sum_{n=0}^{\infty} \frac{(-1)^n (0.5)^{3n+4}}{(3n+4)n!}$$

$$= \frac{(0.5)^4}{4 \cdot 0!} - \frac{(0.5)^7}{7 \cdot 1!} + \frac{(0.5)^{10}}{10 \cdot 2!} - \frac{(0.5)^{13}}{13 \cdot 3!} + \cdots$$

SINCE $b_{n+1} = \frac{(0.5)^{3n+7}}{(3n+7)(n+1)!} < \frac{(0.5)^{3n+4}}{(3n+4)n!} = b_n$

AND $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{(0.5)^{3n+7}}{(3n+7)(n+1)!} = 0$ WE CAN USE THE ALTERNATING SERIES ESTIMATION THM.

NOW $b_3 = \frac{(0.5)^{13}}{13 \cdot 3!} \approx 0.00000157 < 0.000002$

NOW $S_2 = a_0 + a_1 + a_2 = \frac{(0.5)^4}{4 \cdot 0!} - \frac{(0.5)^7}{7 \cdot 1!} + \frac{(0.5)^{10}}{10 \cdot 2!} \approx 0.01455776$

SINCE $\left| \int_0^{0.5} f(x)dx - S_2 \right| < 0.000002$ BY ASET

$\int_0^{0.5} f(x)dx = 0.0146$ WITH |ERROR| < 0.0001

Question 3. *(5 marks)* For the following vector function find the projection of the curve on the yz ~~xz~~-plane. Plot the points for $t = -\pi/2,\ 0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$ on the *xyz*-coordinate system. Use this information to sketch the curve.

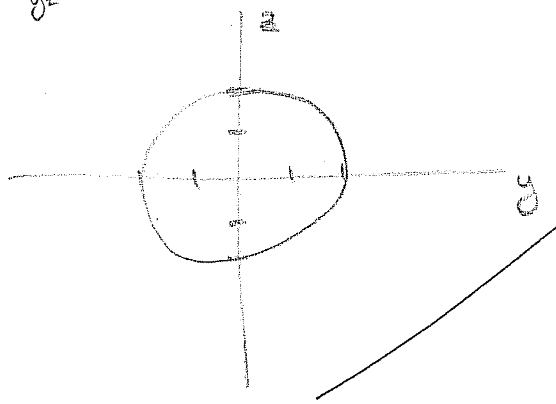
$$\mathbf{r}(t) = \langle t,\ 2\sin t,\ 2\cos t \rangle$$

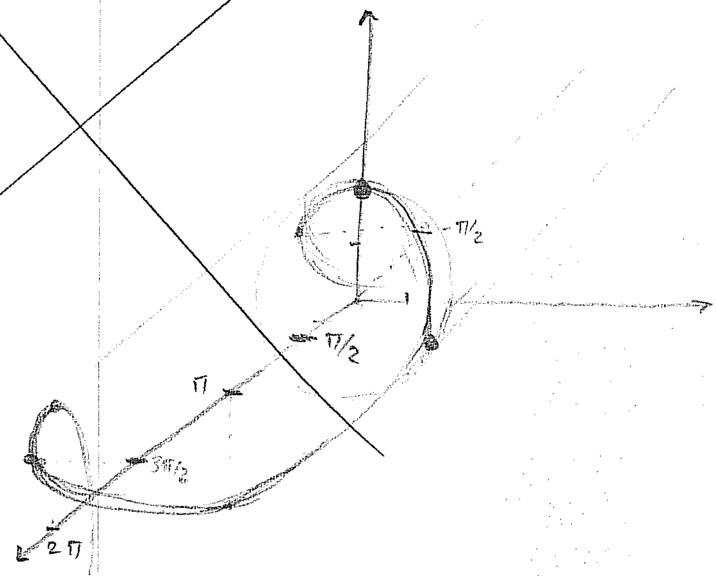
$\vec{r}(-\frac{\pi}{2}) = \langle -\pi/2, -2, 0 \rangle$, $\vec{r}(0) = \langle 0, 0, 2 \rangle$

$\vec{r}(\pi/2) = \langle \pi/2, 2, 0 \rangle$, $\vec{r}(\pi) = \langle \pi, 0, -2 \rangle$, $\vec{r}(\frac{3\pi}{2}) = \langle \frac{3\pi}{2}, -2, 0 \rangle$

$\vec{r}(2\pi) = \langle 2\pi, 0, 2 \rangle$

yz - PLANE PROJECTION





Question 4. *(4 marks)* Find the following limit:

$$L = \lim_{t\to\infty} \left\langle \cos\frac{1}{t},\ \frac{1-2t}{1+5t},\ te^{-2t} \right\rangle = \left\langle \lim_{t\to\infty} \cos\frac{1}{t},\ \lim_{t\to\infty} \frac{1-2t}{1+5t},\ \lim_{t\to\infty} te^{-2t} \right\rangle$$

Now $\lim_{t\to\infty} \cos\frac{1}{t} = \cos(0) = 1$

$$\lim_{t\to\infty} \frac{1-2t}{1+5t} = -\frac{2}{5}$$

$$\lim_{t\to\infty} te^{-2t} = \lim_{t\to\infty} \frac{t}{e^{2t}} \overset{(H)}{=} \lim_{t\to\infty} \frac{1}{2e^{2t}} = 0$$

I.F. $\frac{\infty}{\infty}$

$\therefore\ L = \langle 1, -\frac{2}{5}, 0 \rangle$

Question 5. *(6 marks)* Repramatrize the following curve with respect to arc length starting from the point $(0,1,1)$. At what point on the curve, (a,b,c) is the length of the curve from $(0,1,1)$ to (a,b,c) equal to 1.

$$\mathbf{r}(t) = \langle e^t \sin t,\ e^t \cos t,\ e^t \rangle \qquad \vec{r}\,'(t) = \langle e^t \sin t + e^t \cos t,\ e^t \cos t - e^t \sin t,\ e^t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 + (e^t)^2}$$

$$= \sqrt{e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^t \sin t \cos t + e^{2t}\sin^2 t + e^{2t}\cos^2 t - 2e^t \sin t \cos t + e^{2t}}$$

$$= \sqrt{e^{2t}(1 + 1 + 1)} = e^t\sqrt{3}$$

$$\therefore \quad s(t) = \int_0^t e^u \sqrt{3}\, du = \sqrt{3} e^u \Big|_0^t = \sqrt{3}e^t - \sqrt{3}$$

$$\therefore \quad s = \sqrt{3}e^t - \sqrt{3} \Rightarrow s + \sqrt{3} = \sqrt{3}e^t \Rightarrow \frac{s+\sqrt{3}}{\sqrt{3}} = e^t$$

$$\therefore \quad t = \ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)$$

$$\therefore \quad \vec{r}(t(s)) = \left\langle e^{\ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)} \sin\left[\ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)\right],\ e^{\ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)} \cos\left[\ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)\right],\ e^{\ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)} \right]$$

$$= \left\langle \frac{s+\sqrt{3}}{\sqrt{3}} \sin \ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right),\ \frac{s+\sqrt{3}}{\sqrt{3}} \cos \ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right),\ \frac{s+\sqrt{3}}{\sqrt{3}} \right\rangle$$

THE POINT THAT YIELD ARC LENGTH 1 IS

$$\vec{r}(t(1)) = \left\langle \frac{1+\sqrt{3}}{\sqrt{3}} \sin \ln\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right),\ \frac{1+\sqrt{3}}{\sqrt{3}} \cos \ln\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right),\ \frac{1+\sqrt{3}}{\sqrt{3}} \right\rangle$$

Question 6. *(12 marks)* Let $\mathbf{r}(t) = \frac{1}{2}t^2\vec{i} + \frac{4}{3}t^{3/2}\vec{j} + 2t\vec{k}$

(a) Find the curvature for any t and when $t = 1$.

$\vec{r}\,'(t) = \langle t, 2t^{1/2}, 2\rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{t^2+4t+4}$

$\vec{r}\,''(t) = \langle 1, t^{-1/2}, 0\rangle$ $\qquad = \sqrt{(t+2)^2} = |t+2|$

$$\vec{r}\,'(t)\times\vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & 2t^{1/2} & 2 \\ 1 & t^{-1/2} & 0 \end{vmatrix} = -2t^{-1/2}\vec{i} + (t^{1/2} - 2t^{1/2})\vec{k} - (-2)\vec{j}$$

$$= \langle -2t^{-1/2}, 2, -t^{1/2}\rangle$$

$\therefore |\vec{r}\,'(t)\times\vec{r}\,''(t)| = \sqrt{4t^{-1} + 4 + t}$

$$\kappa(t) = \frac{\sqrt{4t^{-1}+4+t}}{|(t+2)|^3}$$

$$\kappa(1) = \frac{\sqrt{4+4+1}}{(1+2)^3} = \frac{\sqrt{9}}{27} = \frac{1}{9}$$

(b) Find the vectors **T**, **N**, and **B** at the point $\left(\frac{1}{2}, \frac{4}{3}, 2\right)$.

$$\vec{r}'(t) = \langle t,\ 2t^{1/2},\ 2\rangle$$

$$|\vec{r}'(t)| = \sqrt{t^2+4t+4} = \sqrt{(t+2)^2} = t+2$$

$$\therefore\ \vec{T}(t) = \left\langle \frac{t}{t+2}, \frac{2t^{1/2}}{t+2}, \frac{2}{t+2} \right\rangle \Rightarrow \vec{T}(1) = \underline{\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle}$$

$$\vec{T}'(t) = \left\langle \frac{t+2-t}{(t+2)^2}, \frac{t^{-1/2}(t+2) - 2t^{1/2}}{(t+2)^2}, \frac{-2}{(t+2)^2} \right\rangle$$

$$\vec{T}'(1) = \left\langle \frac{2}{9}, \frac{1}{9}, -\frac{2}{9} \right\rangle$$

$$|\vec{T}'(1)| = \sqrt{\frac{4}{81}+\frac{1}{81}+\frac{4}{81}} = \frac{3}{9} = \frac{1}{3}$$

$$\therefore\ N(1) = \frac{\vec{T}'(1)}{|\vec{T}'(1)|} = 3\left\langle \frac{2}{9}, \frac{1}{9}, -\frac{2}{9} \right\rangle = \underline{\left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle}$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{vmatrix} = \left(-\frac{4}{9} - \frac{2}{9}\right)\vec{i} - \left(-\frac{2}{9} - \frac{4}{9}\right)\vec{j} + \left(\frac{1}{9} - \frac{4}{9}\right)\vec{k}$$

$$= \underline{\left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle}$$

(c) Find the equations of the normal plane and the osculating plane of the curve at the point $(\frac{1}{2}, \frac{4}{3}, 2)$.

THE NORMAL PLANE AT $(1/2, 4/3, 2)$ HAS NORMAL VECTOR $\vec{T}(1) = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

$\therefore \quad ax + by + cz + d = 0$

$$\frac{1}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{4}{3} + \frac{2}{3}\cdot 2 + d = 0 \Rightarrow d = \frac{-43}{18}$$

NORMAL PLANE AT $(\frac{1}{2}, \frac{4}{3}, 2)$: $\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z - \frac{43}{18} = 0$

THE OSCULATING PLANE AT $(1/2, 4/3, 2)$ HAS NORMAL VECTOR $\vec{B}(1) = \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle$

$\therefore \quad ax + by + cz + d = 0$

$$-\frac{2}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{4}{3} + \left(-\frac{1}{3}\right)\cdot 2 + d = 0 \Rightarrow d = \frac{1}{9}$$

OSCULATING PLANE AT $(1/2, 4/3, 2)$: $-\frac{2}{3}x + \frac{2}{3}y - \frac{1}{3}z + \frac{1}{9} = 0$

Question 7. *(10 marks)* A particle starting at position $(1,2,0)$ has initial velocity $\mathbf{v}_0 = 3\mathbf{i}+7\mathbf{k}$ and acceleration $\mathbf{a}(t) = 3\mathbf{i}+2t\mathbf{j} - 4t^3\vec{k}$

(a) Find the position vector of the particle.

$\vec{v}(t) = \int \vec{a}(t)dt = \langle 3t+c_1,\ t^2+c_2,\ -t^4+c_3\rangle$

SINCE $\vec{v}(0) = \langle 3,0,7\rangle = \langle c_1, c_2, c_3\rangle$

$\vec{v}(t) = \langle 3t+3,\ t^2,\ -t^4+7\rangle$

$\vec{r}(t) = \langle \frac{3}{2}t^2+3t+k_1,\ \frac{1}{3}t^3+k_2,\ -\frac{1}{5}t^5+7t+k_3\rangle$

$\vec{r}(0) = (1,2,0) = \langle k_1, k_2, k_3\rangle$

$\therefore\ \vec{r}(t) = \langle \frac{3}{2}t^2+3t+1,\ \frac{1}{3}t^3+2,\ -\frac{1}{5}t^5+7t\rangle$

(b) Find the tangetial and normal components of the accelration vector when $t=1$. Show that $\mathbf{a} = a_T\mathbf{T}+a_N\mathbf{N}$ at $t=1$.

$\vec{r}\,'(t) = \vec{v}(t) = \langle 3t+3,\ t^2,\ -t^4+7\rangle \Rightarrow \vec{r}\,'(1) = \langle 6,1,6\rangle$

$\vec{r}\,''(t) = \vec{a}(t) = \langle 3,\ 2t,\ -4t^3\rangle \Rightarrow \vec{r}\,''(1) = \langle 3,2,-4\rangle$

$\vec{r}\,'(1)\cdot\vec{r}\,''(1) = 18+2-24 = -4 \qquad |\vec{r}\,'(1)| = \sqrt{36+1+36} = \sqrt{73}$

$\therefore\ a_T = \dfrac{\vec{r}\,'(1)\cdot\vec{r}\,''(1)}{|\vec{r}\,'(1)|} = \dfrac{-4}{\sqrt{73}}$

$\vec{r}\,'(1)\times\vec{r}\,''(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 1 & 6 \\ 3 & 2 & -4 \end{vmatrix} = -16\vec{i} - (-42)\vec{j} + 9\vec{k}$

$\therefore\ |\vec{r}\,'(1)\times\vec{r}\,''(1)| = \sqrt{256+1764+81} = \sqrt{2101} \qquad \therefore\ a_N = \dfrac{\sqrt{2101}}{\sqrt{73}}$

NOW $\vec{T}(1) = \dfrac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|} = \dfrac{1}{\sqrt{73}}\langle 6,1,6\rangle$

$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$

$$a_T \vec{T} = \frac{-4}{\sqrt{73}} \cdot \frac{1}{\sqrt{73}} \langle 6, 1, 6 \rangle = \frac{-4}{73} \langle 6, 1, 6 \rangle$$

$$a_N \vec{T} = \frac{\sqrt{2101}}{\sqrt{73}} \cdot \frac{73}{\sqrt{1229}} \cdot \frac{1}{\sqrt{73}} \langle -21, -2, -28 \rangle$$

$$= \sqrt{\frac{2101}{1229}} \langle -21, -2, -28 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(3t+3)^2+(t^2)^2+(-t^4+7)^2}$$

$$= \sqrt{9t^2+18t+9+t^4+t^8-14t^4+49}$$

$$= \sqrt{t^8-13t^4+9t^2+18t+58}$$

$$\therefore\ \vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{|\vec{r}\,'(t)|}\langle 3t+3,\ t^2,\ -t^4+7\rangle$$

$$\vec{T}\,'(t) = \frac{d}{dt}\left[\frac{1}{|\vec{r}\,'(t)|}\right]\langle 3t+3,\ t^2,\ -t^4+7\rangle$$

$$+\ \frac{1}{|\vec{r}\,'(t)|}\langle 3,\ 2t,\ -4t^3\rangle$$

~~$= \frac{1}{2}(t^8-13t^4+52t^4+18t+18)$~~

$$= \tfrac{1}{2}(t^8-13t^4+9t^2+18t+58)^{-1/2}\cdot(8t^7-52t^3+18t+18)\cdot\langle 3t+3, t^2, -t^4+7\rangle$$

$$+\ \frac{1}{|\vec{r}\,'(t)|}\langle 3,\ 2t,\ -4t^3\rangle$$

$$\therefore\ \vec{T}\,'(1) = \tfrac{1}{2}(73)^{-1/2}\cdot(-8)\cdot\langle 6,1,6\rangle + \frac{1}{\sqrt{73}}\langle 3,2,-4\rangle$$

$$= \frac{-4}{\sqrt{73}}\langle 6,1,6\rangle + \frac{1}{\sqrt{73}}\langle 3,2,-4\rangle = \left\langle \frac{-21}{\sqrt{73}}, \frac{-2}{\sqrt{73}}, \frac{-28}{\sqrt{73}}\right\rangle$$

$$|\vec{T}\,'(1)| = \frac{1}{73}\sqrt{1229}$$

$$\therefore\ \vec{N}(1) = \frac{73}{\sqrt{1229}}\cdot\frac{1}{\sqrt{73}}\langle -21,\ -2,\ -28\rangle$$

Bonus. *(4 marks)* Show that if a particle moves with constant speed then the velocity and acceleration vectors are ortogonal.