NAME: ____________

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Dawson College: **Calculus III**: 201-BZF-05 S01 October 28, 2011

Test

Question 1. *(5 marks)* Find the limit if it exists, otherwise show that it doesn't exist:

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{2y^2+x^2}$$

ALONG $y=0 \Rightarrow f(\cancel{x},0) = \dfrac{0\cdot \sin^2 x}{0+x^2} = 0 \to 0$ AS $(x,y)\to(0,0)$

ALONG $y=x \Rightarrow f(x,x) = \dfrac{x^2\sin^2 x}{2x^2+x^2} = \dfrac{\sin^2 x}{3} \to 0$ AS $(x,y)\to(0,0)$.

LOOKS LIKE THE LIMIT MIGHT BE 0.

NOW $0 \le \dfrac{y^2\sin^2(x)}{2y^2+x^2} \le \dfrac{y^2\sin^2(x)}{2y^2} = \dfrac{\sin^2(x)}{2}$

NOW $\dfrac{\sin^2 x}{2} \to 0$ AS $(x,y)\to(0,0)$

AND $0\to 0$ AS $(x,y)\to(0,0)$

$\therefore \displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{2y^2+x^2} = 0$ BY SQUEEZE THM.

Question 2. *(5 marks)* Determine the set of points where the following function is continuous.

$$f(x,y) = \begin{cases} -\frac{2xy^3}{x^4+3y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

ALONG $y=0$, $f(x,0) = \frac{0}{x^4} = 0 \to 0$ AS $(x,y) \to (0,0)$

ALONG $y=x$, $f(x,x) = -\frac{2x^4}{x^4+3x^4} = -\frac{2x^4}{4x^4} = -\frac{1}{2} \to \frac{1}{2}$

AS $(x,y) \to (0,0)$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} -\frac{2xy^3}{x^4+3y^4}$ D.N.E.

$\therefore f(0,0) \neq \lim_{(x,y)\to(0,0)} f(x,y)$

$\therefore$ f IS NOT CONTINUOUS AT $(0,0)$

SINCE $f(x,y)$ IS A RATIONAL FUNCTION EXCEPT AT $(0,0)$

IT IS CONTINOUSS EVERYWHERE EXCEPT WHEN $x^4+3y^4=0$.

BUT THIS IS AT $(0,0)$

$\therefore$ f IS CONTINUOUS ON $\{(x,y) \in \mathbb{R} \mid (x,y) \neq (0,0)\}$

Question 3.

(a) *(5 marks)* Find the first partial derivatives f_x, f_y and f_z given:

$$f(x,y,z) = y^{-3/2}\arctan\left(\frac{z}{x}\right) + z\ln(y+e^x)$$

$$f_x = y^{-3/2} \cdot \frac{1}{1+\left(\frac{z}{x}\right)^2} \cdot \left(-\frac{z}{x^2}\right) + z\left(\frac{1}{y+e^x}\right)e^x$$

$$= -\frac{zy^{-3/2}}{x^2+z^2} + \frac{ze^x}{y+e^x}$$

$$f_y = -\frac{3}{2}y^{-5/2}\arctan\left(\frac{z}{x}\right) + z\left(\frac{1}{y+e^x}\right)(1)$$

$$f_z = y^{-3/2} \cdot \frac{1}{1+\left(\frac{z}{x}\right)^2} \cdot \frac{1}{x} + \ln(y+e^x)$$

(b) *(5 marks)* Find f_{xxy} given:

$$f(x,y) = \sin(x^3 - 3y^2)$$

$$f_x = 3x^2\cos(x^3-3y^2)$$

$$f_{xx} = 6x\cos(x^3-3y^2) + 3x^2\left[-\sin(x^3-3y^2) \cdot 3x^2\right]$$

$$= 6x\cos(x^3-3y^2) - 9x^4\sin(x^3-3y^2)$$

$$f_{xxy} = -6x\sin(x^3-3y^2)\cdot(-6y) - 9x^4\cos(x^3-3y^2)\cdot(-6y)$$

Question 4. *(6 marks)* State what it means for a function $f(x,y)$ to be differentiable at a point (a,b) and explain why the function below is differentiable at $(4,3)$. Find the linearization of f at (4,3) and use it to approximate the value of $f(3.92,3.01)$.

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

f IS DIFFERENTIABLE AT (a,b) IF f_x AND f_y ARE CONTINUOUS AT (a,b). AND EXIST NEAR (a,b).

$$f_x = -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot (2x) = \frac{-x}{(x^2+y^2)^{3/2}}$$

$$f_y = -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot 2y = -\frac{y}{(x^2+y^2)^{3/2}}$$

SINCE f_x AND f_y ARE CONTINUOUS AT $(4,3)$ f IS DIFFERENTIABLE AT $(4,3)$

NOW $f_x(4,3) = \frac{-4}{(4^2+3^2)^{3/2}} = \frac{-4}{(\sqrt{25})^3} = \frac{-4}{125}$

$$f_y(4,3) = \frac{-3}{(4^2+3^2)^{3/2}} = -\frac{3}{125}$$

$$f(4,3) = \frac{1}{\sqrt{4^2+5^2}} = \frac{1}{5}$$

$$\therefore L(x,y) = f(4,3) + f_x(4,3)(x-4) + f_y(4,3)(y-3)$$

$$= \frac{1}{5} + \left(-\frac{4}{125}\right)(x-4) + \left(-\frac{3}{125}\right)(y-3)$$

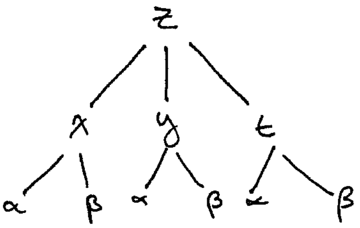
$$= \frac{1}{5} - \frac{4}{125}x + \frac{16}{125} - \frac{3}{125}y + \frac{9}{125}$$

$$= -\frac{4}{125}x - \frac{3}{125}y + \frac{2}{5}$$

$$f(3.92, 3.01) \approx L(3.92, 3.01) = \frac{-4}{125}(3.92) - \frac{3}{125}(3.01) + \frac{2}{5}$$

Question 5.

(a) *(4 marks)* Use chain rule to find an expression for $\frac{\partial z}{\partial \alpha}$ and $\frac{\partial z}{\partial \beta}$ given $z = te^{1-x^2y}$ and $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$, and $t = t(\alpha, \beta)$ are functions of α and β.



$$\frac{\partial z}{\partial \alpha} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial \alpha}$$
$$= \left(te^{1-x^2y}\right)(-2xy)\frac{\partial x}{\partial \alpha} + \left(te^{1-x^2y}\right)(-x^2)\frac{\partial y}{\partial \alpha} + e^{1-x^2y}\frac{\partial t}{\partial \alpha}$$

$$\frac{\partial z}{\partial \beta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \beta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \beta} + \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial \beta}$$
$$= \left(te^{1-x^2y}\right)(-2xy)\frac{\partial x}{\partial \beta} + \left(te^{1-x^2y}\right)(-x^2)\frac{\partial y}{\partial \beta} + e^{1-x^2y}\frac{\partial t}{\partial \beta}$$

(b) *(4 marks)* Using part **(a)**, given $x = \alpha^2\beta$, $y = \alpha \cancel{\cos} \beta^2$, and $t = 3\alpha\beta^{\frac{4}{3}}$ find $\frac{\partial z}{\partial \alpha}$ and $\frac{\partial z}{\partial \beta}$ when $\alpha = 2$ and $\beta = -1$

$\therefore x = 2^2(-1) = -4 \qquad y = 2 \cdot (-1)^2 = 2 \qquad t = 3 \cdot 2\,(-1)^{4/3} = 6$

$$\therefore \frac{\partial z}{\partial \alpha} = \left(te^{1-x^2y}\right)(-2xy)(2\alpha\beta) + \left(te^{1-x^2y}\right)(-x^2)\,\beta^2 + e^{1-x^2y}\,3\beta^{4/3}$$
$$= 6e^{1-(-2)^2(2)}(-2(-4)(2))(2 \cdot 2 \cdot (-1)) + 6e^{1-(-2)^2(2)}(-(-4)^2) \cdot (-1)^2 + e^{1-(-2)^2(2)} \cdot 3(-1)^{4/3} = -384e^{-31} - 96e^{-31} + 3e^{-31} = -477e^{-31}$$

$$\frac{\partial z}{\partial \alpha} = \left(te^{1-x^2y}\right)(-2xy)(\alpha^2) + \left(te^{1-x^2y}\right)(-x^2)(2\alpha\beta) + e^{1-x^2y}(4\alpha\beta$$
$$= 6e^{1-(-2)^2(2)}(-2(-4)(2))(2^2) + \left(6e^{1-(-2)^2(2)}\right)(-(-4)^2)(2(-2)(-1)) + e^{1-(-2)^2(2)}(4(2)(-1)^{1/3}) = (384 + 384 - 8)e^{-3}$$

Question 6.

(a) *(5 marks)* Find the directional derivative of $f(x,y,z) = x^3z - yx^2 + z^2$ at $(2,-1,1)$ in the direction of $\mathbf{v} = \langle 3,-1,2 \rangle$.

$f_x = 3x^2z - 2yx, \quad f_y = -x^2, \quad f_z = x^3 + 2z$

$\nabla f = \langle 3x^2z - 2yx, -x^2, x^3 + 2z \rangle$

$\nabla f(2,-1,1) = \langle 3(2)^2(1) - 2(-1)(2), -(2)^2, (2)^3 + 2(1) \rangle = \langle 16, -4, 10 \rangle$

$\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{9+1+4}} \langle 3, -1, 2 \rangle = \left\langle \dfrac{3}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}} \right\rangle$

$\therefore D_{\vec{u}} f(2,-1,1) = \nabla f(2,-1,1) \cdot \vec{u}$

$= \langle 16, -4, 10 \rangle \cdot \left\langle \dfrac{3}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}} \right\rangle$

$= \dfrac{48}{\sqrt{14}} + \dfrac{4}{\sqrt{14}} + \dfrac{20}{\sqrt{14}} = \dfrac{72}{\sqrt{14}}$

(b) *(2 marks)* Find the maximum rate of change of f at $(2,-1,1)$ and the direction in which it occurs.

MAXIMUM RATE OF CHANGE: $|\nabla f(2,-1,1)| = \sqrt{16^2 + (-4)^2 + 10^2}$

$= \sqrt{256 + 16 + 100}$

$= \sqrt{372}$

IT OCCURS IN THE DIRECTION OF $\nabla f(2,-1,1) = \langle 16, -4, 10 \rangle$

Question 7. *(6 marks)* Locate all relative extrema and saddle points of $f(x,y) = 4xy - x^4 - y^4$

$f_x = 4y - 4x^3 = 0 \qquad f_y = 4x - 4y^3 = 0$

$\Downarrow$

$y = x^3 \implies 4x - 4(x^3)^3 = 4x - 4x^6 = 0$

$\implies 4x(1 - x^8) = 0$

$\implies x = 0, \pm 1$

$\therefore$ $4y - 4(0) = 0 \implies y = 0 \quad \therefore (0,0)$ IS A C.P.

$4y - 4(1)^3 = 0 \implies y = 1 \quad \therefore (1,1)$ IS A C.P.

$4y - 4(-1)^3 = 0 \implies y = -1 \quad \therefore (-1,-1)$ IS A C.P.

$D = f_{xx} f_{yy} - (f_{xy})^2 = (-12x^2)(-12y^2) - (4)^2$

$D(0,0) = 0 \cdot 0 - 16 = -16 < 0 \quad \therefore$ f HAS A SADDLE POINT AT $(0,0)$

$D(1,1) = (-12)(-12) - 16 = 128 > 0$ AND $f_x(1,1) = -12 < 0$

$\therefore$ $f(1,1) = 4 - 1 - 1 = 2$ IS A RELATIVE MAXIMUM. (AT $(1,1)$)

$D(-1)(-1) = (-12)(-12) - 16 = 128 > 0$ AND $f_x(-1,-1) = -12 < 0$

$\therefore$ $f(-1,-1) = 4 - 1 - 1 = 2$ IS A RELATIVE MAXIMUM (AT $(-1,-1)$).

Question 8. *(6 marks)* Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

$d = \sqrt{(\cancel{x}-0)^2 + (y-0)^2 + (z-0)^2} \Rightarrow d^2 = x^2+y^2+z^2$

$x^2 = 5+yz \Rightarrow d^2 = 5+yz+y^2+z^2 = f(y,z)$

$\therefore f_y = z+2y=0, \quad f_z = y+2z=0$

$\Rightarrow z = -2y \Rightarrow y+2(-2y)=0 \Rightarrow -3y=0 \Rightarrow y=0$

$\therefore z+2(0)=0 \Rightarrow z=0$

NOW, SOME POINT ON $x^2-yz=5$ WILL YIELD THE MINIMUM DISTANCE AND SO IT MUST OCCUR AT A CRITIC POINT SINCE $(0,0)$ IS THE ONLY C.P. IT MUST YIELD THE MIN.

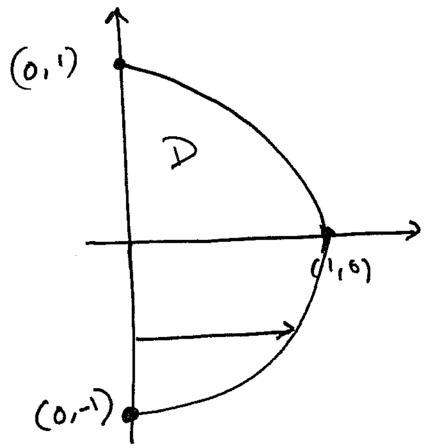
$x^2 = 5+(0)(0) = 5 \Rightarrow x = \pm\sqrt{5}$

$\therefore$ THE MINIMUM DISTANCE IS $d = \sqrt{(-\sqrt{5})^2+0^2+0^2} = \sqrt{5}$

AT $(\sqrt{5},0,0)$ AND $(-\sqrt{5},0,0)$

Question 9. *(6 marks)* Let D be the region in the xy-plane enclosed by $x = 0$ and $x = \sqrt{1-y^2}$. Find

$$\iint_D xy^2\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} xy^2\,dx\,dy$$



$$= \int_{-1}^{1} \left[\frac{1}{2}x^2y^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1} \left[\frac{1}{2}(1-y^2)y^2 - 0\right] dy = I$$

$D = \{(x,y) \mid -1 \le y \le x,\ 0 \le x \le \sqrt{1-y^2}\}$ (TYPE II)

$$I = \frac{1}{2}\int_{-1}^{1} y^2 - y^4\,dy = \frac{1}{2}\left[\frac{1}{3}y^3 - \frac{1}{5}y^5\right]_{-1}^{1}$$

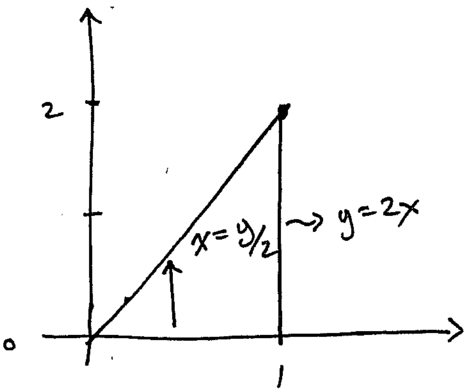
$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{2}{15}$$

Question 10. *(5 marks)* Evaluate the following iterated integral:

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy \quad = \iint_D e^{x^2}\, dA$$

WHERE $D = \{(x,y) \mid 0 \le y \le 2,\ \frac{y}{2} \le x \le 1\}$



$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2x\}$

$$\therefore \int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy = \iint_D e^{x^2}\, dA = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 \left[y e^{x^2} \right]_{y=0}^{y=2x} dx = \int_0^1 2x e^{x^2}\, dx$$

LET $u = x^2$

$du = 2x\, dx$

$\frac{du}{2x} = dx$

$$= \int_{x=0}^{x=1} e^u\, du = e^u \Big|_{x=0}^{x=1} = e^{x^2} \Big|_0^1$$

$$= e - 1$$