1. Evaluate $\lim_{x\to 3}\frac{\sqrt{x^2+7}-4}{x-3}$

$$\lim_{x\to 3}\frac{\sqrt{x^2+7}-4}{x-3}\frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4}=\lim_{x\to 3}\frac{x^2+7-16}{(x-3)\left(\sqrt{x^2+7}+4\right)}=\lim_{x\to 3}\frac{x^2-9}{(x-3)\left(\sqrt{x^2+7}+4\right)}$$

$$\lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)\left(\sqrt{x^2+7}+4\right)}=\lim_{x\to 3}\frac{(x+3)}{\left(\sqrt{x^2+7}+4\right)}=\frac{3}{4}$$

2. Use the definition of continuity to determine whether $f(x)$ is continuous at $x=1$.

$$f(x)=\begin{cases}2x-1 & if\ x<1\\ 0 & if\ x=1\\ x^2 & if\ x>1\end{cases}$$

$f(1)=0$ $\qquad\qquad$ $\lim_{x\to 1^-} 2x-1=1$

Therefore, $f(1)\neq\lim_{x\to 1} f(x)$, thus $f(x)$ is not continuous at $x=1$.

3. Use the limit definition of the derivative to find $f'(x)$ for: $f(x)=x^2-3x$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-3(x+h)-\left(x^2-3x\right)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}=\lim_{h\to 0}\frac{2xh+h^2-3h}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{h(2x+h-3)}{h}=\lim_{h\to 0}(2x+h-3)=2x-3$$

4. Differentiate each of the following. Do <u>not</u> simplify the answer.

a. $f(x)=e^{-x^2}(x^3+1)$ $\qquad$ $f'(x)=-2xe^{-x^2}(x^3+1)+3x^2e^{-x^2}$

b. $s(t)=\left(\frac{t^2}{t+1}\right)^5+3\arctan 5t$

$$s'(t)=5\left(\frac{t^2}{t+1}\right)^4\frac{2t(t+1)-t^2}{(t+1)^2}+3\frac{5}{1+(5t)^2}$$

c. $f(x)=\sin(\tan 3x)$ $\qquad$ $f'(x)=3\cos(\tan 3x)\sec^2 3x$

d. $y=\ln\sqrt[3]{x^2+x}$ $\qquad$ Use log properties.

$y=\frac{1}{3}\ln\left(x^2+x\right)$ $\qquad$ $y'=\frac{1}{3}\frac{2x+1}{x^2+x}$

5. Find $\frac{dy}{dx}$ by implicit differentiation given that $3x^4 - 2x^2y^3 - y^2 = -17$

$$12x^3 - 4xy^3 - 6x^2y^2\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$-6x^2y^2\frac{dy}{dx} - 2y\frac{dy}{dx} = -12x^3 + 4xy^3$$

$$\left(-6x^2y^2 - 2y\right)\frac{dy}{dx} = -12x^3 + 4xy^3 \qquad \frac{dy}{dx} = \frac{-12x^3 + 4xy^3}{-6x^2y^2 - 2y}$$

6. Find an equation of the line tangent to $f(x) = 2x\ln x$ at $x = e$.

$$f'(x) = 2\ln x + 2x\frac{1}{x} = 2\ln x + 2$$

$$m = 2\ln e + 2 = 4$$

$$f(e) = 2e\ln e = 2e \qquad y - 2e = 4(x - e)$$

Or: $2e = 4e + b \Rightarrow b = -2e$ $\qquad y = 4x - 2e$

7. Use logarithmic differentiation to find y' given that

a. $y = \frac{x^2\sqrt{2x+7}}{(4x-1)^4}$ $\qquad \ln y = \ln\frac{x^2\sqrt{2x+7}}{(4x-1)^4}$

$$\ln y = \ln x^2\sqrt{2x+7} - \ln(4x-1)^4 = \ln x^2 + \ln\sqrt{2x+7} - \ln(4x-1)^4$$

$$\ln y = 2\ln x + \frac{1}{2}\ln(2x+7) - 4\ln(4x-1)$$

$$\frac{y'}{y} = \frac{2}{x} + \frac{1}{2}\frac{2}{2x+7} - 4\frac{4}{4x-1} \qquad y' = \left(\frac{2}{x} + \frac{1}{2x+7} - \frac{16}{4x-1}\right)y$$

$$y' = \left(\frac{2}{x} + \frac{1}{2x+7} - \frac{16}{4x-1}\right)\frac{x^2\sqrt{2x+7}}{(4x-1)^4}$$

b. $y = (\cos x)^x$

$$\ln y = \ln(\cos x)^x \qquad \ln y = x\ln(\cos x)$$

$$\frac{y'}{y} = \ln(\cos x) + x\frac{-\sin x}{\cos x} \qquad y' = \left(\ln(\cos x) - x\tan x\right)y$$

$$y' = \left(\ln(\cos x) - x\tan x\right)(\cos x)^x$$

8. Find the value(s) of x where the following functions have horizontal tangent lines:

a. $f(x) = (3x+6)^4(4x-12)^3$

$$f'(x) = 12(3x+6)^3(4x-12)^3 + 12(3x+6)^4(4x-12)^2$$

$$f'(x) = 0 \Rightarrow 12(3x+6)^3(4x-12)^3 + 12(3x+6)^4(4x-12)^2 = 0$$

$$12(3x+6)^3(4x-12)^2\left[(4x-12)+(3x+6)\right]=0$$

$$12(3x+6)^3(4x-12)^2(7x-6)=0 \qquad x=-2 \vee x=3 \vee x=\frac{6}{7}$$

b. $f(x)=xe^{3x}$ $\qquad f'(x)=e^{3x}+3xe^{3x}$

$$f'(x)=0 \Rightarrow e^{3x}+3xe^{3x}=0 \Rightarrow e^{3x}(1+3x)=0 \Rightarrow x=-\frac{1}{3}$$

9. Given $f(x)=x^4-4x^3+2$:

a. Find all critical points.
b. Find the intervals in which $f(x)$ is increasing, decreasing.
c. Find all the maximum and minimum points.
d. Find the intervals in which $f(x)$ is concave up and concave down.
e. Find the points of inflection, if any.
f. Use the information to sketch the graph of $f(x)$.

a. $f'(x)=4x^3-12x^2$

$$f'(x)=0 \Rightarrow 4x^3-12x^2=0 \Rightarrow 4x^2(x-3)=0 \Rightarrow x=0 \vee x=3$$

b.

x	-∞		0		3		∞
$f'(x)$		-	0	-	0	+	
$f(x)$		dec	|	dec	|	inc	

$f(x)$ is increasing on $]3,\infty[$ and decreasing on $]-\infty,0[\cup]0,3[$

c. $f(3)=81-108+2=-25$ $f(x)$ local min at $(3,-25)$. $f(x)$ has no local max.

d. $f''(x)=12x^2-24x$

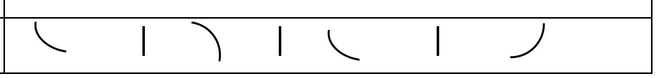
$$f''(x)=0 \Rightarrow 12x^2-24x=0 \Rightarrow 12x(x-2)=0 \Rightarrow x=0 \vee x=2$$

x	-∞		0		2		∞
$f''(x)$		+	0	-	0	+	
$f(x)$		CU	|	CD	|	CU	

$f(x)$ is concave down on $]0,2[$ and concave up on $]-\infty,0[\cup]2,\infty[$

e. inflection points at: (0,2) and (2,-14)

f.

x	-∞		0		2		3		∞
$f'(x)$		-	0		-		0	+	
$f''(x)$		+	0	-	0		+		
$f(x)$									
			2		-14		-25		
			I.P		I.P		l-min		

Graph here.

10. Find the absolute extrema of $f\left(x\right)=x^3-12x$ on the interval [-3,1].

$f'\left(x\right)=3x^2-12$

$f'\left(x\right)=0\Rightarrow 3x^2-12=0\Rightarrow 3\left(x^2-4\right)=0$

$3(x+2)(x-2)=0\Rightarrow x=-2\vee x=2$

$x=2\notin[-3,1]$

$f\left(-3\right)=9$ $\quad f\left(-2\right)=16$ $\quad f\left(1\right)=-11$

Absolute max at (-2,16) and absolute min at (1,-11)

11. Find $f\left(x\right)$ if $f''\left(x\right)=2x+3$ $f'\left(-1\right)=2$ and $f\left(0\right)=3$

$\int f''(x)dx=\int\left(2x+3\right)dx=x^2+3x+C$

$f'\left(-1\right)=2\Rightarrow 1-3+C=2\Rightarrow C=4$

$f'(x)=x^2+3x+4$

$\int f'(x)dx=\int\left(x^2+3x+4\right)dx=\frac{1}{3}x^3+\frac{3}{2}x^2+4x+C$

$f\left(0\right)=3\Rightarrow C=3$

$f(x)=\frac{1}{3}x^3+\frac{3}{2}x^2+4x+3$

12. N/A

13. A rectangular box with a square base and no top is to have a total surface area of 108 square meters. Find the dimensions of the box with the largest possible volume.

Maximize $V=x^2y$

Subject to: $x^2+4xy=108$

$x^2+4xy=108\Rightarrow 4xy=108-x^2\Rightarrow y=\frac{108-x^2}{4x}$

$V(x)=x^2\frac{108-x^2}{4x}=27x-\frac{1}{4}x^3$

$V'(x)=27-\frac{3}{4}x^2$

$V'(x)=0\Rightarrow 27-\frac{3}{4}x^2=0\Rightarrow\frac{3}{4}x^2=27\Rightarrow x^2=36\Rightarrow x=\pm 6$

Reject $x=-6$

$V''(x)=-\frac{6}{4}x\Rightarrow V''(6)<0$ Therefore x=6 maximizes the volume.

Replacing x=6 in the subject to equation, yields y=3. Thus the dimension of the box maximizing the volume is: 6X6X3

14. N/A

15. Evaluate the following integrals.

a. $I = \int \left(\frac{x^{-4}}{2} - 7x + 4\pi \right) dx = -\frac{x^{-3}}{6} - \frac{7x^2}{2} + 4\pi x + C$

b. N/A

c. $I = \int \left(\frac{x^5 - 2x^{-1} + 5}{\sqrt{x}} \right) dx = \int \left(x^{9/2} - 2x^{-3/2} + 5x^{-1/2} \right) dx$

$I = \int \left(x^{9/2} - 2x^{-3/2} + 5x^{-1/2} \right) dx = \frac{2}{11} x^{11/2} + 4x^{-1/2} + 10x^{1/2} + C$