

Last Name: SOLUTIONS

First Name: _____

Student ID: _____

Test 1

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**.

Question 1. (4 marks) Simplify the following expression writing your final answer with positive exponents:

$$\begin{aligned}
 & \left(\frac{9x^8z^{-1}}{x^{-3}y^2} \right)^{\frac{1}{2}} \cdot \left(\frac{xy^2z}{x^{-1}y^3} \right)^{-2} = \frac{9^{1/2}(x^8)^{1/2}(z^{-1})^{1/2}}{(x^{-3})^{1/2}(y^2)^{1/2}} \cdot \frac{(x)^{-2}(y^2)^{-2}z^{-2}}{(x^{-1})^{-2}(y^3)^{-2}} \\
 & = \frac{3x^4z^{-1/2}}{x^{-3/2}y} \cdot \frac{x^{-2}y^{-4}z^{-2}}{x^2y^{-6}} = \frac{3x^4x^{3/2}}{z^{1/2}y} \cdot \frac{y^6}{x^2x^2y^4z^2} \\
 & = \frac{3x^{11/2}y^6}{x^4y^5z^{5/2}} = \frac{3x^{3/2}y}{z^{5/2}}
 \end{aligned}$$

Question 2. (4 marks) Simplify the following expression writing your final answer with positive exponents:

$$\frac{4x^2(3x^2+1)^{-\frac{1}{3}} - (3x^2+1)^{\frac{2}{3}}}{x+1} = \frac{\frac{4x^2}{(3x^2+1)^{1/3}} - (3x^2+1)^{2/3}}{x+1}$$

$$\begin{aligned}
 & = \frac{\frac{4x^2}{(3x^2+1)^{1/3}} - \frac{(3x^2+1)}{(3x^2+1)^{1/3}}}{x+1} = \frac{\frac{4x^2 - (3x^2+1)}{(3x^2+1)^{1/3}}}{x+1} = \frac{\frac{4x^2 - 3x^2 - 1}{(3x^2+1)^{1/3}}}{x+1} \\
 & = \frac{\frac{x^2 - 1}{(3x^2+1)^{1/3}}}{x+1} = \frac{(x^2 - 1)}{(3x^2+1)^{1/3}} \cdot \frac{1}{x+1} \\
 & = \frac{(x+1)(x-1)}{(3x^2+1)^{1/3}} \cdot \frac{1}{x+1} = \frac{x-1}{(3x^2+1)^{1/3}}
 \end{aligned}$$

Question 3. Evaluate the following:

(a) (1 mark) $\cos\left(\frac{7\pi}{3}\right) = \frac{1}{2}$

(b) (1 mark) $\csc\left(-\frac{5\pi}{4}\right) = \frac{1}{\sin\left(-\frac{5\pi}{4}\right)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}}$ or $-\sqrt{2}$

(d) (2 mark) ~~$\arcsin\left(\tan\left(\frac{5\pi}{6}\right)\right)$~~

(f) (2 mark) Find θ on $0 \leq \theta \leq 2\pi$ given $\sec \theta = \frac{2}{\sqrt{3}}$ $\Rightarrow \frac{1}{\cos \theta} = \frac{2}{\sqrt{3}}$

$\Rightarrow \cos \theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = \frac{\pi}{6}, \frac{11\pi}{6}$

Question 4. Evaluate the following limits:

$$(a) (2 \text{ marks}) \lim_{x \rightarrow 7^+} \frac{2 - \sqrt{x-3}}{x+5} = \frac{2 - \sqrt{7-3}}{7+5} = \frac{2 - \sqrt{4}}{12} = \frac{0}{12} = 0 \quad \text{Q.E.D.}$$

$$(b) (4 \text{ marks}) \lim_{x \rightarrow 3} \frac{x^2 + 2x - 15}{x^3 - 27} = \lim_{x \rightarrow 3} \frac{(x+5)(x-3)}{(x-3)(x^2 + 3x + 9)}$$

$$= \lim_{x \rightarrow 3} \frac{x+5}{x^2 + 3x + 9} = \frac{3+5}{(3)^2 + 3(3) + 9} = \frac{8}{27}$$

$$(c) (4 \text{ marks}) \lim_{x \rightarrow -5} \frac{x+5}{6 - \sqrt{41+x}} = \lim_{x \rightarrow -5} \frac{x+5}{6 - \sqrt{41+x}} \cdot \frac{6 + \sqrt{41+x}}{6 + \sqrt{41+x}}$$

$$= \lim_{x \rightarrow -5} \frac{(x+5)(6 + \sqrt{41+x})}{36 + 6\sqrt{41+x} - 6\sqrt{41+x} - (41+x)} = \lim_{x \rightarrow -5} \frac{(x+5)(6 + \sqrt{41+x})}{-5 - x}$$

$$= \lim_{x \rightarrow -5} \frac{(x+5)(6 + \sqrt{41+x})}{-(x+5)} = \lim_{x \rightarrow -5} - (6 + \sqrt{41+x})$$

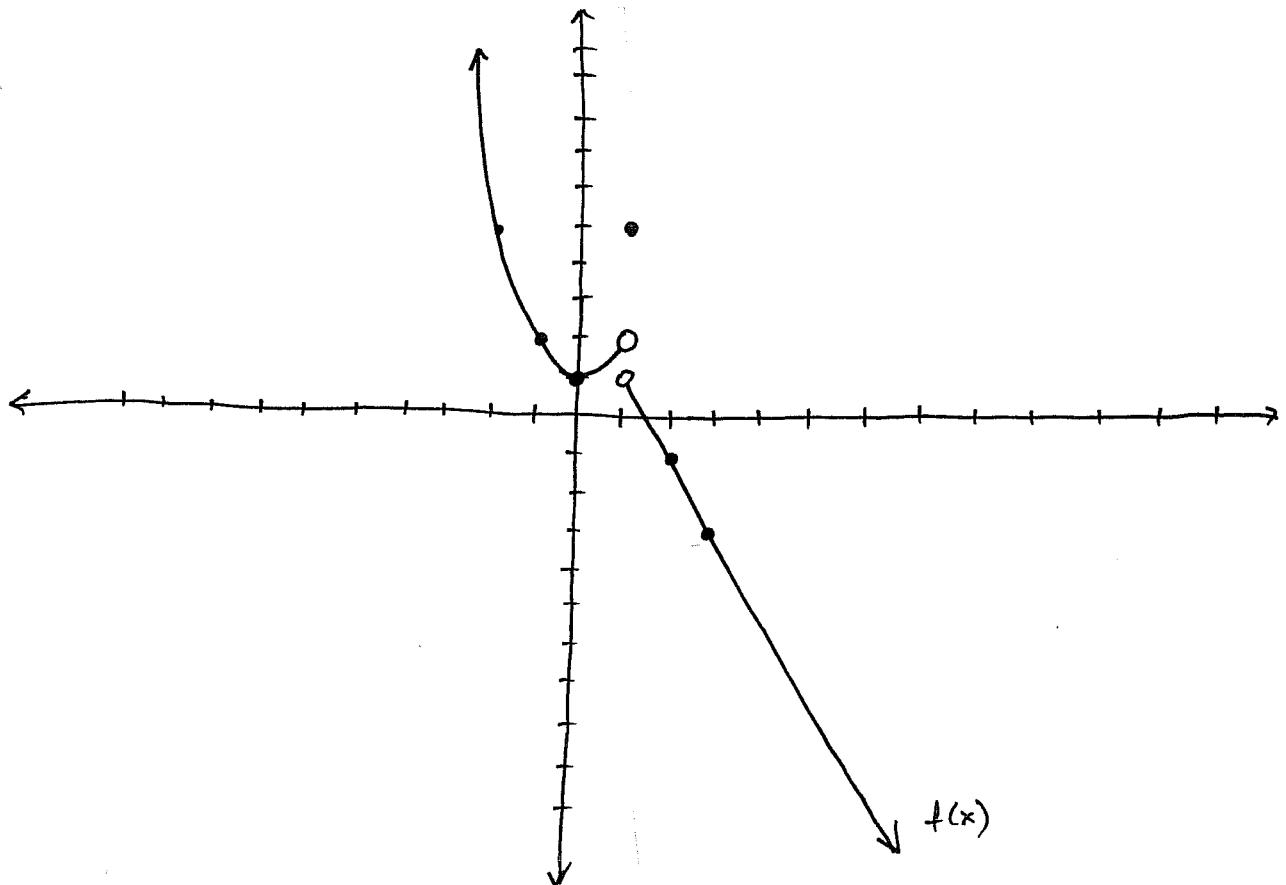
$$= -6 - \sqrt{41+5} = -6 - 6 = -12$$

$$\begin{aligned}
 \text{(d) (4 marks)} \lim_{x \rightarrow -\infty} \frac{-4x^4 + 2x^2 - x + 2}{2x^4 - 9x^3 + 2} &= \lim_{x \rightarrow -\infty} \frac{-4x^4/x^4 + 2x^2/x^4 - x/x^4 + 2/x^4}{2x^4/x^4 - 9x^3/x^4 + 2/x^4} \\
 &= \lim_{x \rightarrow -\infty} \frac{-4 + 2/x^2 - 1/x^3 + 2/x^4}{2 - 9/x + 2/x^4} = \frac{-4 + 0 - 0 + 0}{2 - 0 + 0} = -2
 \end{aligned}$$

$$\begin{aligned}
 \text{(d) (4 marks)} \lim_{x \rightarrow \infty} \frac{5x^5 + 2x^3 - 2x}{x^3 - 9x + 2} &= \lim_{x \rightarrow \infty} \frac{5x^5/x^3 + 2x^3/x^3 - 2x/x^3}{x^3/x^3 - 9x/x^3 + 2/x^3} \\
 &= \lim_{x \rightarrow \infty} \frac{5x^2 + 2 - 2/x^2}{1 - 9/x^2 + 2/x^3} = \infty \quad (\text{D.N.E.})
 \end{aligned}$$

Question 5. (4 marks) Graph the following function and use the graph to determine where the function is continuous. If the function is discontinuous at a point $x = a$ state which condition of continuity it fails.

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 5 & \text{if } x = 1 \\ -2x + 3 & \text{if } x > 1 \end{cases}$$



f IS NOT CONTINUOUS AT $x=1$ SINCE IT FAILS CONDITION 2

($\lim_{x \rightarrow 1^-} f(x)$ D.N.E. SINCE $\lim_{x \rightarrow 1^-} f(x) = 2 \neq 1 = \lim_{x \rightarrow 1^+} f(x)$).

f IS CONTINUOUS EVERYWHERE ELSE (EVERYWHERE EXCEPT $x=1$).

Question 6. (5 marks) For which values of x is the following function continuous? (Do not graph this function). Clearly explain your reasoning.

$$f(x) = \begin{cases} \frac{x^2-4}{x-1} & \text{if } x < -2 \\ x+2 & \text{if } x \geq -2 \end{cases}$$

when $x < -2$, $f(x) = \frac{x^2-4}{x-1}$ which is a rational function that is continuous except when $x-1=0 \Leftrightarrow x=1$ but since $x < -2$ this doesn't have any effect on f . so f is continuous when $x < -2$.

when $x > -2$, $f(x) = x+2$ a polynomial $\therefore f$ is continuous when $x > -2$.

AT $x = -2$.

$$1) f(-2) = -2+2 = 0$$

$$2) \lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{x^2-4}{x-1} = \frac{(-2)^2-4}{-2+1} = \frac{0}{1} = 0$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} x+2 = -2+2 = 0$$

$$\therefore \lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^+} f(x) \quad \text{AND SO} \quad \lim_{x \rightarrow -2} f(x) \text{ EXIST (AND }= 0\text{)}$$

$$3) \lim_{x \rightarrow -2} f(x) = 0 = f(-2)$$

so f is continuous at $x = -2$

$\therefore f$ is continuous everywhere.

Question 7. (6 marks) Let $f(x) = x^2 + 3x - 2$.

(a) Find $f'(x)$ using the limit definition of the derivative.

(b) Find the point(s) on the graph of f where the tangent line is horizontal.

(c) Find the tangent line to the curve at $x = 3$.

$$\begin{aligned} \text{a) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 3(x+h) - 2] - [x^2 + 3x - 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - 2 - x^2 - 3x + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h + 3)}{h} \\ &= \lim_{h \rightarrow 0} (2x + h + 3) = 2x + 0 + 3 = 2x + 3 \end{aligned}$$

b) HORIZONTAL TANGENT LINE \Leftrightarrow SLOPE OF TANGENT = 0 $\Leftrightarrow f'(x) = 0$
 $\therefore 2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$

$$y = f\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) - 2 = \frac{9}{4} - \frac{9}{2} - 2 = -\frac{9}{4} - 2 = -\frac{17}{4}$$

\therefore THE POINT WHERE THE TANGENT LINE IS HORIZONTAL IS $(-\frac{3}{2}, -\frac{17}{4})$

c) SLOPE OF TANGENT LINE AT $x = 3$ = $f'(3) = 2(3) + 3 = 9 = m$

$$y = f(3) = (3)^2 + 3(3) - 2 = 16$$

$$\therefore y = mx + b$$

$$16 = 9(3) + b$$

$$\begin{aligned} 16 - 27 &= b \\ -11 &= b \end{aligned}$$

\therefore THE EQUATION OF THE TANGENT LINE AT $x = 3$ IS

$$y = 9x - 11.$$