

Last Name: SOLUTIONS

First Name: _____

Student ID: _____

Test 3

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**.

Question 1.(a) (4 marks) Find $\frac{dy}{dx}$ given $(x+y)^{\frac{3}{2}} - 2xy = -2$ (b) (2 marks) Find the equation of the tangent line to the graph of y at the point $(1, 3)$

$$a) \frac{d}{dx} [(x+y)^{\frac{1}{2}}] - \frac{d}{dx} [2xy] = \frac{d}{dx} [-2]$$

$$\frac{3}{2}(x+y)^{\frac{1}{2}}(1+\frac{dy}{dx}) - 2y - 2x\frac{dy}{dx} = 0$$

$$\frac{3}{2}(x+y)^{\frac{1}{2}} + \frac{3}{2}(x+y)^{\frac{1}{2}}\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} = 0$$

$$\frac{3}{2}(x+y)^{\frac{1}{2}} \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y - \frac{3}{2}(x+y)^{\frac{1}{2}}$$

$$\frac{dy}{dx} \left[\frac{3}{2}(x+y)^{\frac{1}{2}} - 2x \right] = 2y - \frac{3}{2}(x+y)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{2y - \frac{3}{2}(x+y)^{\frac{1}{2}}}{\frac{3}{2}(x+y)^{\frac{1}{2}} - 2x}$$

$$b) m = \left. \frac{dy}{dx} \right|_{(1,3)} = \frac{2(3) - \frac{3}{2}(1+3)^{\frac{1}{2}}}{\frac{3}{2}(1+3)^{\frac{1}{2}} - 2(1)} = \frac{6 - \frac{3}{2}(2)}{\frac{3}{2}(2) - 2} = \frac{3}{1} = 3$$

$$y = mx + b$$

$$3 = 3(1) + b$$

$$0 = b$$

$$\therefore y = 3x$$

Question 2. (5 marks) Suppose the wholesale price of a certain brand of medium-sized eggs p (in dollars/carton) is related to the weekly supply x (in thousand of cartons) by the equation

$$625p^2 - x^2 = 100$$

If the 25 000 cartons of eggs are available at the beginning of a certain week and the price is falling at the rate of $\$0.02$ per carton per week, at what rate is the supply falling? (ROUND TO THE NEAREST UNIT)

$$\frac{dp}{dt} = -0.02, \quad x = 25 \Rightarrow 625p^2 - (25)^2 = 100 \\ 625p^2 = 725 \Rightarrow p^2 = 1.16 \Rightarrow p = \sqrt{1.16}$$

$$\frac{d}{dt}[625p^2] - \frac{d}{dt}[x^2] = \frac{d}{dt}[100]$$

$$1250p \frac{dp}{dt} - 2x \frac{dx}{dt} = 0$$

$$1250(\sqrt{1.16})(-0.02) - 2(25)\frac{dx}{dt} = 0$$

$$-50 \frac{dx}{dt} = 25\sqrt{1.16}$$

$$\frac{dx}{dt} = -0.5385165$$

∴ THE SUPPLY IS INCREASING AT A RATE OF 53 UNITS PER WEEK.

Question 3. (5 marks) Use the second derivative test to find the relative extrema of

$$f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - 4x^2 + 5$$

$$f'(x) = x^3 - 2x^2 - 8x = 0$$

$$x(x^2 - 2x - 8) = 0$$

$$x(x+2)(x-4) = 0$$

$$\therefore x = -2, 0, 4$$

$$f''(x) = 3x^2 - 4x - 8$$

$$f''(-2) = 3(-2)^2 - 4(-2) - 8 = 12 > 0 \Rightarrow f(-2) = -\frac{5}{3} \text{ IS A RELATIVE MINIMUM}$$

$$f''(0) = -8 < 0 \Rightarrow f(0) = 5 \text{ IS A RELATIVE MAXIMUM}$$

$$f''(4) = 3(4)^2 - 4(4) - 8 = 24 > 0 \Rightarrow f(4) = \frac{-113}{3} \text{ IS A RELATIVE MINIMUM}$$

Question 4. (4 marks) Find all vertical and horizontal asymptotes (indicate which is which) of the function

$$f(x) = \frac{-4x^2 - 5}{x^2 - 4x + 4}$$

$$x^2 - 4x + 4 = 0 \Rightarrow (x-2)(x-2) = 0 \Rightarrow x=2$$

$$P(2) = -4(2)^2 - 5 = -21 \neq 0$$

$\therefore x=2$ IS A VERTICAL ASYMPTOTE

$$\lim_{x \rightarrow \infty} \frac{-4x^2 - 5}{3x^2 - 4x - 4} = \lim_{x \rightarrow \infty} \frac{-4 - 5/x^2}{3 - 4/x - 4/x^2} = \frac{-4 - 0}{3 - 0 - 0} = -\frac{4}{3}$$

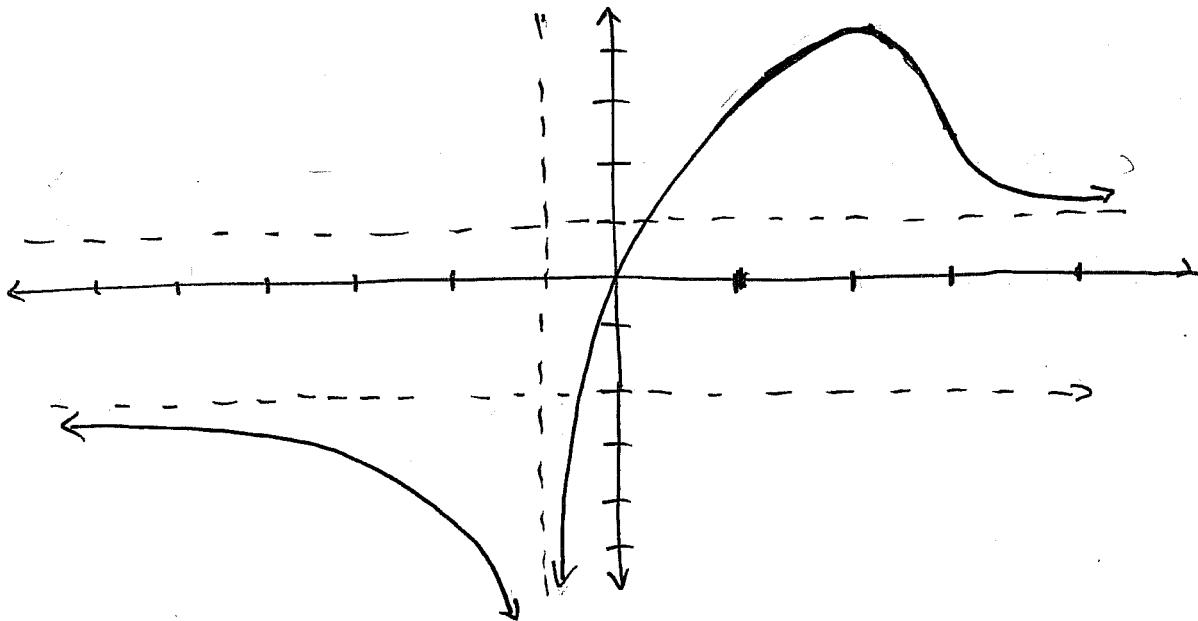
$\therefore y = -\frac{4}{3}$ IS A HORIZONTAL ASYMPTOTE

$$\lim_{x \rightarrow -\infty} \frac{-4x^2 - 5}{3x^2 - 4x - 4} = \lim_{x \rightarrow -\infty} \frac{-4 - 5/x^2}{3 - 4/x - 4/x^2} = \frac{-4 - 0}{3 - 0 - 0} = -\frac{4}{3}$$

$\therefore y = -\frac{4}{3}$ IS A HORIZONTAL ASYMPTOTE.

Question 5. Using the following graph of $f(x)$ to find the following information:

- (a) (1 marks) Horizontal and vertical asymptotes of $f(x)$ (indicate which is which).
- (b) (1 marks) Intervals where f is increasing and intervals where f is decreasing.
- (c) (1 marks) Any relative extrema.
- (d) (1 marks) Intervals where f is concave upward and intervals where f is concave downward.
- (e) (1 marks) Any inflection points.



- a) H.A. $y=1$, V.A. $x=-1$
- b) f is increasing on $(-1, 2)$ and f is decreasing on $(-\infty, -1)$ and $(2, \infty)$.
- c) $(2, 4)$ is a relative max.
- d) f is concave upward on $(3, \infty)$ and concave downward on $(-\infty, -1)$ and $(1, 3)$
- e) $(3, 2)$ is an I.P.

Question 6. Given $f(x) = x^4 - 4x^3$ find:

(a) (1 marks) The domain of $f(x)$.

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(b) (1 marks) The x and y -intercepts.

$$\begin{aligned}x\text{-int: } y &= 0 \\0 &= x^4 - 4x^3 \\0 &= x^3(x-4) \\x &= 0, 4 \\\therefore (0,0), (4,0)\end{aligned}$$

$$\begin{aligned}y\text{-int: } x &= 0 \\y &= 0^4 - 4(0)^3 = 0 \\\therefore (0,0)\end{aligned}$$

(c) (3 marks) Intervals where f is increasing and intervals where f is decreasing.

$$f'(x) = 4x^3 - 12x^2$$

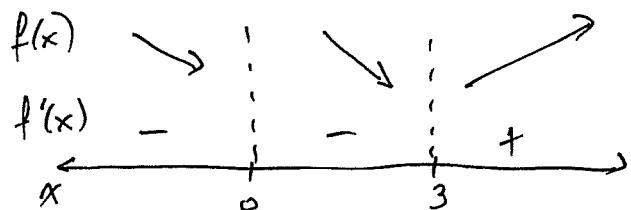
$$\begin{aligned}f'(x) &= 0 \\4x^2(x-3) &= 0 \\x=0, x=3\end{aligned}$$

$f'(x) \text{ D.N.C.}$

POLYNOMIAL

TEST POINTS

$$\begin{aligned}f'(-1) &= -16 < 0 \\f'(1) &= -8 < 0 \\f'(4) &= 64 > 0\end{aligned}$$



$\therefore f$ is INCREASING ON $(3, \infty)$
 f is DECREASING ON $(-\infty, 0)$ AND $(0, 3)$

(d) (1 marks) Any relative extrema.

$$f(3) = (3)^4 - 4(3)^3 = -27 \text{ IS A RELATIVE MINIMUM.}$$

(e) (3 marks) Intervals where f is concave upward and intervals where f is concave downward.

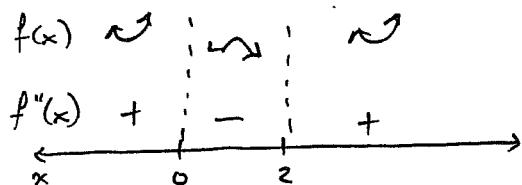
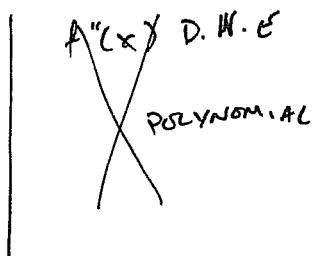
$$f'(x) = 12x^2 - 24x$$

$$f''(x) = 0$$

$$12x^2 - 24x = 0$$

$$12x(x-2) = 0$$

$$x = 2, 0$$



TEST POINTS

$$f''(-1) = 36 > 0$$

$$f''(1) = -12 < 0$$

$$f''(3) = 36 > 0$$

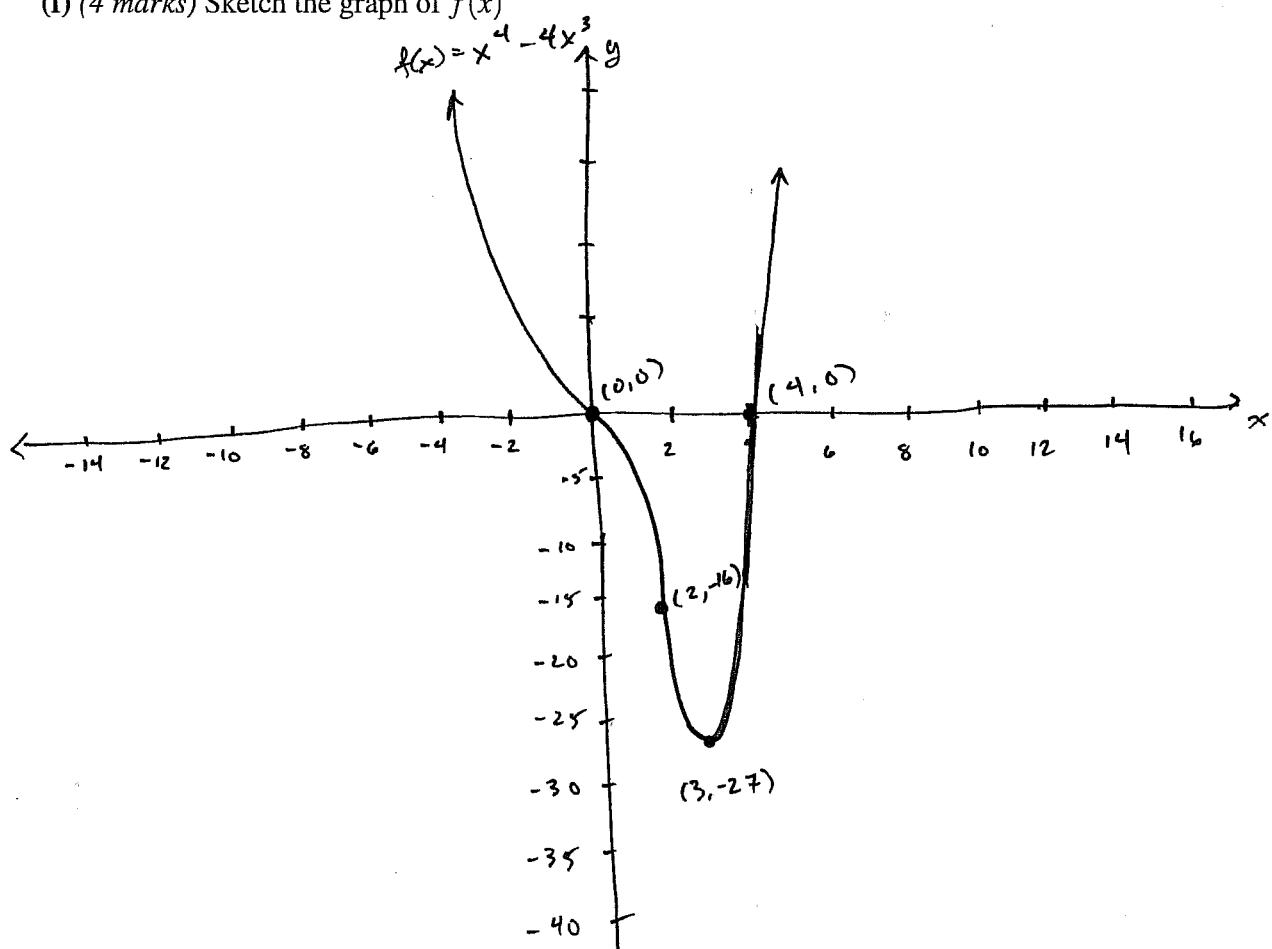
$\therefore f$ IS CONCAVE UPWARD ON $(-\infty, 0)$ AND $(2, \infty)$
 f IS CONCAVE DOWNWARD ON $(0, 2)$

(f) (1 marks) Any inflection points.

$$f(0) = 0 \rightarrow \text{AN INFLECTION POINT} \Rightarrow (0, 0)$$

$$f(2) = -16 \rightarrow \text{AN INFLECTION POINT} \Rightarrow (2, -16)$$

(f) (4 marks) Sketch the graph of $f(x)$



Question 7. (4 marks) Use the method developed in class for optimizing a function on a closed interval to find the absolute maximum and absolute minimum value of $f(x) = x^3 - 4x^2 + 4x + 1$ on the interval $[1, 5]$.

$$f'(x) = 3x^2 - 8x + 4$$

$$= 3x^2 - 6x - 2x + 4$$

$$= 3x(x-2) - 2(x-2)$$

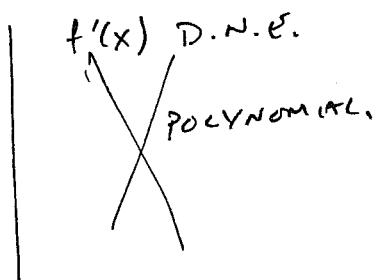
$$= (3x-2)(x-2)$$

$$f''(x) = 0$$

$$(3x-2)(x-2) = 0$$

$$x \cancel{\times} \frac{2}{3} \quad x=2$$

NOT IN $[1, 5]$



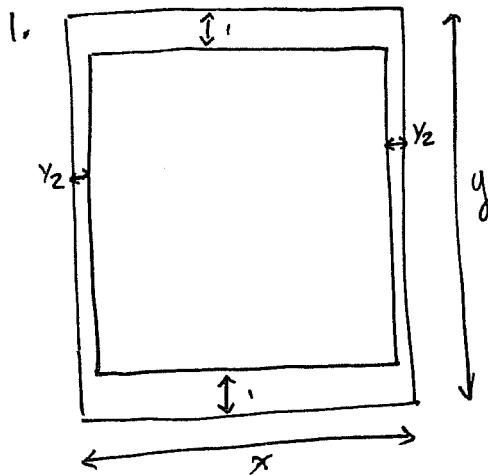
$$\underline{\text{C.P.}} \quad f(2) = (2)^3 - 4(2)^2 + 4(2) + 1 = 1 \quad \leftarrow \text{ABSOLUTE MINIMUM}$$

END POINTS

$$f(1) = (1)^3 - 4(1)^2 + 4(1) + 1 = 2$$

$$f(5) = (5)^3 - 4(5)^2 + 4(5) + 1 = 46 \quad \leftarrow \text{ABSOLUTE MAXIMUM.}$$

Question 8. (6 marks) A book designer has decided that the pages of a book should have 1 inch margins at the top and bottom and $\frac{1}{2}$ inch margins on the sides. She further stipulated that each page (including margins) should have an area of 50 inches². Use the four step procedure outlined in class determine the page dimensions that will result in the maximum printed area on the page.



1. $A = (x-1)(y-2)$

2. $x y = 50 \Rightarrow y = \frac{50}{x}$

$\therefore A(x) = (x-1)\left(\frac{50}{x} - 2\right)$

$= 50 - \frac{50}{x} - 2x + 2$

$= 52 - \frac{50}{x} - 2x$

RESTRICTIONS

$x > 1$

\therefore THE DOMAIN OF $A(x)$ IS $(1, \infty)$

4. $A'(x) = \frac{50}{x^2} - 2$

$A'(x) = 0$

$\frac{50}{x^2} - 2 = \frac{50 - 2x^2}{x^2} = 0$

$50 - 2x^2 = 0$

$25 = x^2$

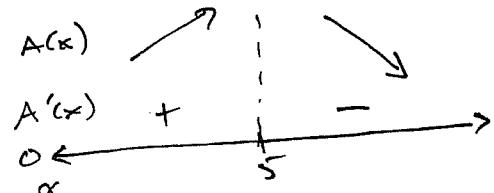
$x = -5, 5$

NOT IN DOMAIN

$A'(x)$ D.N.E.

$x = 0$

A NOT IN DOMAIN



TEST POINTS.

$x = 1 \quad A'(1) = 50 - \frac{50}{1^2} - 2 = 30$

$x = 6 \quad A'(6) = \frac{50}{(6)^2} - 2 = -\frac{11}{18} < 0$

$\therefore x=5$ YIELDS ABSOLUTE MAXIMUM.

\therefore THE DIMENSIONS OF THE PAGE ARE $x=5$ in BY $y=10$ in