Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S1: Fall 2011

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Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate using the definition of the definite integral

$$\int_1^2 -3x^2+2x-1\,dx.$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[-3\left(1+\frac{i}{n}\right)^2 + 2\left(1+\frac{i}{n}\right) - 1\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[-3 - \frac{6i}{n} - \frac{3i^2}{n^2} + 2 + \frac{2i}{n} - 1\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[-2 - \frac{4i}{n} - \frac{3i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\sum_{i=1}^{n} -2 - \sum_{i=1}^{n} \frac{4i}{n} - \sum_{i=1}^{n} \frac{3i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[-2n - \frac{4}{n}\sum_{i=1}^{n} i - \frac{3}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[-2n - \frac{\overset{2}{\cancel{4}}}{\cancel{n}}\frac{\cancel{n}(n+1)}{\cancel{2}} - \frac{\cancel{3}}{n^{\cancel{2}}}\frac{\cancel{n}(n+1)(2n+1)}{\underset{2}{\cancel{6}}}\right]$$

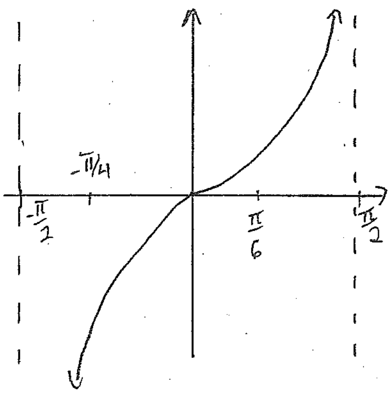
$$= \lim_{n\to\infty} \left[\frac{-2\cancel{n}}{\cancel{n}} - \frac{2(n+1)}{n} - \frac{(n+1)}{n}\cdot\frac{(2n+1)}{2n}\right]$$

$$= -2 - 2 - 1$$

$$= -5$$

Question 2. *(5 marks)* Evaluate the definite integral:

$$\int_{-\pi/4}^{\pi/6} |\tan\theta|\, d\theta = \int_{-\frac{\pi}{4}}^{0} |\tan\theta|\, d\theta + \int_{0}^{\pi/6} |\tan\theta|\, d\theta$$



$$= \int_{-\frac{\pi}{4}}^{0} -\tan\theta\, d\theta + \int_{0}^{\pi/6} \tan\theta\, d\theta$$

$$= \Big[\ln|\cos\theta|\Big]_{-\frac{\pi}{4}}^{0} + \Big[-\ln|\cos\theta|\Big]_{0}^{\pi/6}$$

$$= \underbrace{\ln|\cos 0|}_{0} - \ln|\cos(-\tfrac{\pi}{4})| + \Big[-\ln|\cos\tfrac{\pi}{6}| + \overbrace{\ln|\cos 0|}^{0}\Big]$$

$$= -\ln\left|\frac{1}{\sqrt{2}}\right| - \ln\left|\frac{\sqrt{3}}{2}\right|$$

$$= -\ln\left(\frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}}\right) = -\ln\left(\frac{2}{\sqrt{6}}\right) = \ln\left(\frac{\sqrt{6}}{2}\right)$$

Question 3. *(5 marks)* Evaluate the indefinite integral:

$$\int 2(z^2+z)e^{4z^3+6z^2}\, dz = \int e^u \frac{du}{6}$$

$u = 4z^3 + 6z^2$

$du = (12z^2 + 12z)dz$

$du = 6 \cdot 2(z^2+z)dz$

$\frac{du}{6} = 2(z^2+z)\,dz$

$$= \frac{1}{6}\int e^u du$$

$$= \frac{1}{6}e^u + C$$

$$= \frac{1}{6}e^{4z^3+6z^2} + C$$

Question 4. *(5 marks)* Find the average value of the function

$$f(x) = (3x^2 + 1)\arctan x$$

on the interval $[0, 1]$.

$$\text{average value of function} = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{1-0}\int_0^1 (3x^2+1)\arctan x\,dx$$

$u = \arctan x \quad du = \frac{1}{1+x^2}dx$ $= \int_0^1 (3x^2+1)\arctan x\,dx$

$v = x^3+x \quad dv = 3x^2+1\,dx$ $= \left[uv\right]_0^1 - \int_0^1 v\,du$

$$= \left[(x^3+x)\arctan x\right]_0^1 - \int_0^1 \frac{x^3+x}{x^2+1}dx$$

$$= (1^3+1)\arctan 1 - (0^3+0)\arctan 0 - \int_0^1 \frac{x\cancel{(x^2+1)}}{\cancel{(x^2+1)}}dx$$

$$= 2\arctan 1 - \int_0^1 x\,dx$$

$$= \frac{2\pi}{4} - \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{\pi}{2} - \frac{1}{2}$$

$$= \frac{\pi - 1}{2}$$

Question 5. *(5 marks)* Evaluate the expression:

$$\frac{d}{dx}\left[\underbrace{\int_{2x}^{\csc 3x} u(\tan u)^u\,du}_{h(x)}\right]$$

$$h(x) = \int_{2x}^{\csc 3x} u(\tan u)^u\,du$$

$$= \int_{2x}^{0} u(\tan u)^u\,du + \int_0^{\csc 3x} u(\tan u)^u\,du$$

$$= -\int_0^{2x} u(\tan u)^u\,du + \int_0^{\csc 3x} u(\tan u)^u\,du$$

$$= -f(g_1(x)) + f(g_2(x))$$

where $f(x) = \int_0^x u(\tan u)^u\,du$ $\quad f'(x) = x(\tan x)^x$ by 2^{nd} FTC

$g_1(x) = 2x$ $\quad g_1'(x) = 2$

$g_2(x) = \csc 3x$ $\quad g_2'(x) = -\csc 3x \cot 3x\,(3)$

$$h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$$

$$= -(2x)(\tan(2x))^{2x}\cdot 2 + \csc 3x\,(\tan \csc 3x)^{\csc 3x}(-3)\csc 3x \cot 3x$$

Question 6. *(5 marks)* Evaluate the indefinite integral:

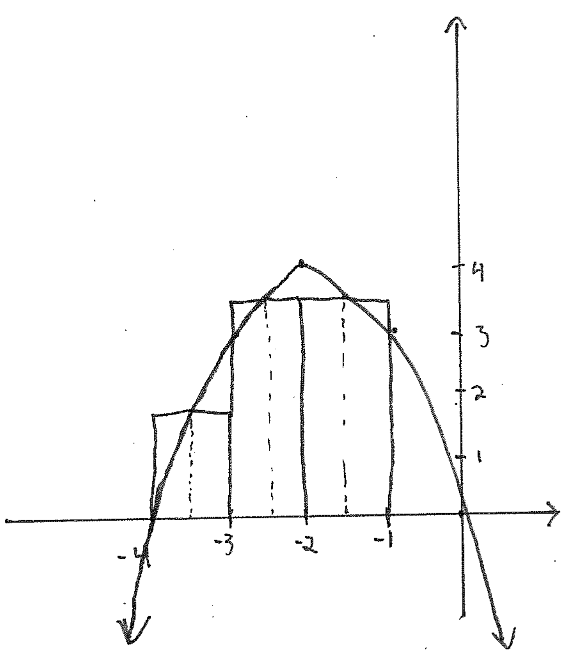
$$\int x \sec 2x \tan 2x \, dx \qquad = uv - \int v \, du$$

$u = x \qquad du = dx \qquad = \frac{x\sec 2x}{2} - \int \frac{\sec 2x}{2} dx$

$v = \frac{\sec 2x}{2} \qquad dv = \sec 2x \tan 2x \, dx$

$$= \frac{x \sec 2x}{2} - \frac{1}{4} \ln|\sec 2x + \tan 2x| + C$$

Question 7. *(5 marks)* Estimate the area under the graph of $f(x) = 4 - (x+2)^2$ from $x = -4$ to $x = -1$ using three rectangles and using midpoints. Sketch the curve and the approximating rectangles.



$\Delta x = \frac{b-a}{n} = \frac{-4-(-1)}{3} =$

$= \frac{-1-(-4)}{3}$

$= 1$

$x_i = a + i\Delta x = -4 + i$

$x_0 = -4$

$x_1 = -4 + 1 = -3$

$x_2 = -4 + 2 = -2$

$x_3 = -4 + 3 = -1$

$x_1^* = \frac{x_0 + x_1}{2} = \frac{-7}{2}$

$x_2^* = \frac{x_1 + x_2}{2} = \frac{-5}{2}$

$x_3^* = \frac{x_2 + x_3}{2} = -\frac{3}{2}$

Area $\approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x$

$= f(\frac{-7}{2}) + f(\frac{-5}{2}) + f(\frac{-3}{2})$

$= 4 - (\frac{-7}{2} + 2)^2 + 4 - (\frac{-5}{2} + 2)^2 + 4 - (\frac{-3}{2} + 2)^2$

$= 4 - (\frac{-3}{2})^2 + 4 - (\frac{-1}{2})^2 + 4 - (\frac{1}{2})^2 = 12 - \frac{9}{4} - \frac{1}{4} - \frac{1}{4} = 12 - \frac{11}{4}$

$= \frac{37}{4}$

Question 8. *(5 marks)* Prove: If $f(x)$ is an odd integrable function on $[-a,a]$ then

$$\int_{-a}^{a} f(x)\,dx = 0$$

$$\text{LHS} = \int_{-a}^{a} f(x)dx$$

$$= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{-a} -f(x)dx + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{-a} f(-x)dx + \int_{0}^{a} f(x)dx$$

$u = -x \quad u(0) = 0$

$du = -dx \quad u(a) = -(-a) = a$

$$= \int_{0}^{a} f(u)(-du) + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx$$

$$= 0$$

Question 9. *(5 marks)* If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_{0}^{a} f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_{0}^{a} f''(x)g(x)\,dx$$

$$\int_{0}^{a} f(x)g''(x)dx = \left[uv\right]_{0}^{a} - \int_{0}^{a} v\,du$$

$u = f(x) \quad du = f'(x)dx$

$v = g'(x) \quad dv = g''(x)dx$

$$= \left[f(x)g'(x)\right]_{0}^{a} - \int_{0}^{a} g'(x)f'(x)dx$$

$$= f(a)g'(a) - f(0)g'(0) - \int_{0}^{a} g'(x)f'(x)dx$$

$$= f(a)g'(a) - \left[\left[uv\right]_{0}^{a} - \int_{0}^{a} v\,du\right]$$

$u = f'(x) \quad du = f''(x)dx$

$v = g(x) \quad dv = g'(x)dx$

$$= f(a)g'(a) - \left[\left[f'(x)g(x)\right]_{0}^{a} - \int_{0}^{a} g(x)f''(x)dx\right]$$

$$= f(a)g'(a) - \left[\left[f'(a)g(a) - f'(0)g(0)\right] - \int_{0}^{a} g(x)f''(x)dx\right]$$

$$= f(a)g'(a) - f'(a)g(a) - \int_{0}^{a} g(x)f''(x)dx$$

Bonus Question. *(3 marks)*
Evaluate:

$$\lim_{h\to 0} \frac{\frac{\int_{\pi}^{x+h} \arctan(t)\,dt - \int_{\pi}^{x} \arctan(t)\,dt}{h}}{\cos h}$$

$$= \frac{\lim_{h\to 0}\left[\int_{\pi}^{x+h} \arctan(t)\,dt - \int_{\pi}^{x} \arctan(t)\,dt\right]}{\lim_{h\to 0} \cos h}$$

$$= \frac{\frac{d}{dx}\left[\int_{\pi}^{x} \arctan t\,dt\right]}{1}$$

$= \arctan x$ by 2^{nd} FTC.