Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S1: Fall 2011

Name: Y.Lamontagne
Student ID: ____________

Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Evaluate the definite integral

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2(2x)\cos^2(2x)\, dx.$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \frac{(1-\cos 4x)}{2}\frac{(1+\cos 4x)}{2}\,dx$$

$$= \frac{1}{4}\int_{\pi/12}^{\pi/8} 1-\cos^2 4x\,dx$$

$$= \frac{1}{4}\int_{\frac{\pi}{12}}^{\pi/8} \sin^2 4x\,dx$$

$$= \frac{1}{4}\int_{\pi/12}^{\pi/8} \frac{1-\cos 8x}{2}\,dx$$

$$= \frac{1}{8}\int_{\pi/12}^{\pi/8} 1-\cos 8x\,dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 8x}{8}\right]_{\pi/12}^{\pi/8}$$

$$= \frac{1}{8}\left[\frac{\pi}{8} - \frac{\sin \pi}{8}\right] - \frac{1}{8}\left[\frac{\pi}{12} - \frac{\sin\frac{2\pi}{3}}{8}\right]$$

$$= \frac{1}{8}\left[\frac{\pi}{8}\right] - \frac{1}{8}\left[\frac{\pi}{12} - \frac{\frac{\sqrt{3}}{2}}{8}\right]$$

$$= \frac{\pi}{64} - \frac{\pi}{96} - \frac{\sqrt{3}}{128}$$

$$= \frac{\pi}{192} + \frac{\sqrt{3}}{128}$$

Question 2. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{x^2}{(9-4x^2)^{3/2}}\,dx$$

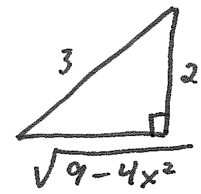
$2x = 3\sin\theta$

$x = \frac{3}{2}\sin\theta$

$dx = \frac{3}{2}\cos\theta\, d\theta$

$\frac{2x}{3} = \sin\theta$

$\arcsin\left(\frac{2x}{3}\right) = \theta$



So $\tan\theta = \frac{2x}{\sqrt{9-4x^2}}$

$$= \int \frac{x^2}{(3^2-(2x)^2)^{3/2}}\,dx$$

$$= \int \frac{\left(\frac{3}{2}\sin\ \right)^2}{(3^2-(3\sin\ )^2)^{3/2}}\ \frac{3}{2}\cos\ d$$

$$= \int \frac{\frac{9}{4}\sin^2\theta}{(9-9\sin^2\ )^{3/2}}\ \frac{3}{2}\cos\ d\theta$$

$$= \frac{9}{4}\cdot\frac{3}{2}\int \frac{\sin^2\theta}{(9(1-\sin^2\theta))^{3/2}}\cos\theta\, d\theta$$

$$= \frac{27}{8}\int \frac{\sin^2\theta}{(9\cos^2\theta)^{3/2}}\cos\theta\, d\theta$$

$$= \frac{27}{8}\int \frac{\sin^2\theta}{27\cos^{3}\theta}\cos\theta\, d\theta$$

$$= \frac{27}{8}\cdot\frac{1}{27}\int \tan^2\theta\, d\theta$$

$$= \frac{1}{8}\int \sec^2\theta - 1\, d\theta$$

$$= \frac{1}{8}\left[\tan\theta - \theta\right] + C$$

$$= \frac{1}{8}\left[\frac{2x}{\sqrt{9-4x^2}} - \arcsin\left(\frac{2x}{3}\right)\right] + C$$

Question 3. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{x^2-x+6}{x(x^2+3)}\,dx$$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$\frac{(x^2-x+6)\cancel{(x)}\cancel{(x^2+3)}}{\cancel{x}\cancel{(x^2+3)}} = \frac{A\cancel{x}(x^2+3)}{\cancel{x}} + \frac{(Bx+C)x\cancel{(x^2+3)}}{\cancel{x^2+3}}$$

$$x^2-x+6 = A(x^2+3) + (Bx+C)x$$

Let $x=0$

$$0^2-0+6 = A(0^2+3) + (B(0)+C)0$$
$$6 = 3A$$
$$2 = A$$

Let $x=1$

$$1^2-1+6 = A(1^2+3) + (B(1)+C)(1)$$
$$6 = 4A + B+C$$
$$6 = 4(2) + B+C$$
$$-2 = B+C \quad ①$$

Let $x=-1$

$$(-1)^2-(-1)+6 = A((-1)^2+3) + (B(-1)+C)(-1)$$
$$8 = 4A + B - C$$
$$4 = 4(2) + B - C$$
$$C = B \quad ②$$

Sub ② into ① $\quad -2 = B+B$

$$-1 = B \qquad \therefore C=-1$$

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{2}{x} + \frac{-x-1}{x^2+3}\,dx$$

$$= \int \frac{2}{x}\,dx - \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

$$= 2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C$$

Question 4. *(5 marks)* Evaluate the improper integral or show it diverges:

$$\int_5^{\infty} \frac{1}{(x-1)\sqrt{x^2-2x-3}}\,dx$$

$$x^2-2x-3$$
$$= x^2-2x+1-1-3$$
$$= (x-1)^2-4$$

$$= \lim_{b\to\infty} \int_5^b \frac{1}{(x-1)\sqrt{x^2-2x-3}}\,dx$$

$u = x-1$ $\quad u(b) = b-1$

$du = dx$ $\quad u(\ ) = 5-1 = 4$

$$= \lim_{b\to\infty} \int_5^b \frac{1}{(x-1)\sqrt{(x-1)^2-4}}\,dx$$

$$= \lim_{b\to\infty} \int_4^{b-1} \frac{1}{u\sqrt{u^2-4}}\,du$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\operatorname{arcsec}\frac{u}{2}\right]_4^{b-1}$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\underbrace{\operatorname{arcsec}\frac{(b-1)}{2}}_{\to \pi/2} - \overbrace{\frac{1}{2}\operatorname{arcsec}\frac{4}{2}}^{\frac{1}{2}\cdot\frac{\pi}{3}}\right]$$

$$= \frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{3}$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$= \frac{\pi}{12}$$

Question 5. *(5 marks)* Evaluate the improper integral or show it diverges:

$$\int_0^1 \frac{\ln x}{\sqrt[3]{x^2}}\, dx$$

$$= \lim_{a \to 0^+} \int_a^1 \frac{\ln x}{x^{2/3}}\, dx$$

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = 3x^{1/3} \qquad dv = \frac{1}{x^{2/3}}\,dx$

$$= \lim_{a \to 0^+} \left[[uv]_a^1 - \int_a^1 v\,du \right]$$

$$= \lim_{a \to 0^+} \left[\left[3x^{1/3} \ln x\right]_a^1 - \int_a^1 \frac{3x^{1/3}}{x}\, dx \right]$$

$$= \lim_{a \to 0^+} \left[3\,1^{1/3} \ln 1 - 3a^{1/3} \ln a - \int_a^1 \frac{3}{x^{2/3}}\, dx \right]$$

$$= \lim_{a \to 0^+} \left[-3a^{1/3} \ln a - \left[9x^{1/3}\right]_a^1 \right]$$ I.F. $0 \cdot \infty$

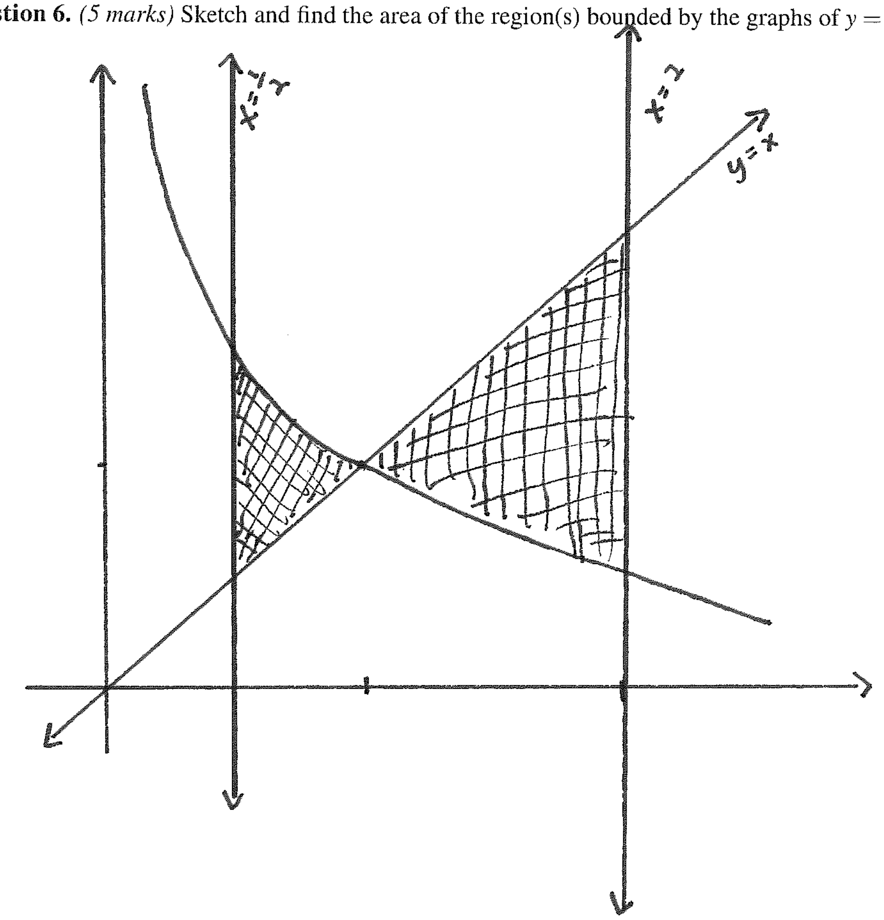
$$= \lim_{a \to 0^+} \left[\frac{-3 \ln a}{a^{-1/3}} - \left[\left[9 \cdot 1^{1/3} - 9\cancel{a^{1/3}}^{\,0}\right]\right] \right]$$

$$= \lim_{a \to 0^+} \left[\frac{-3\, 1/a}{-1/3\, a^{-4/3}} - 9 \right]$$ by $\hat{H}$

$$= \lim_{a \to 0^+} \left[9\cancel{a^{1/3}}^{\,0} - 9 \right]$$

$$= -9$$

Question 6. *(5 marks)* Sketch and find the area of the region(s) bounded by the graphs of $y = x$ and $y = \frac{1}{x^2}$ between $x = \frac{1}{2}$ and $x = 2$.



intersection of two curves

$$x = \frac{1}{x^2}$$

$$x^3 = 1$$

$$x = 1$$

$$\text{Area} = \int_{\frac{1}{2}}^{1} \frac{1}{x^2} - x\,dx + \int_{1}^{2} x - \frac{1}{x^2}\,dx$$

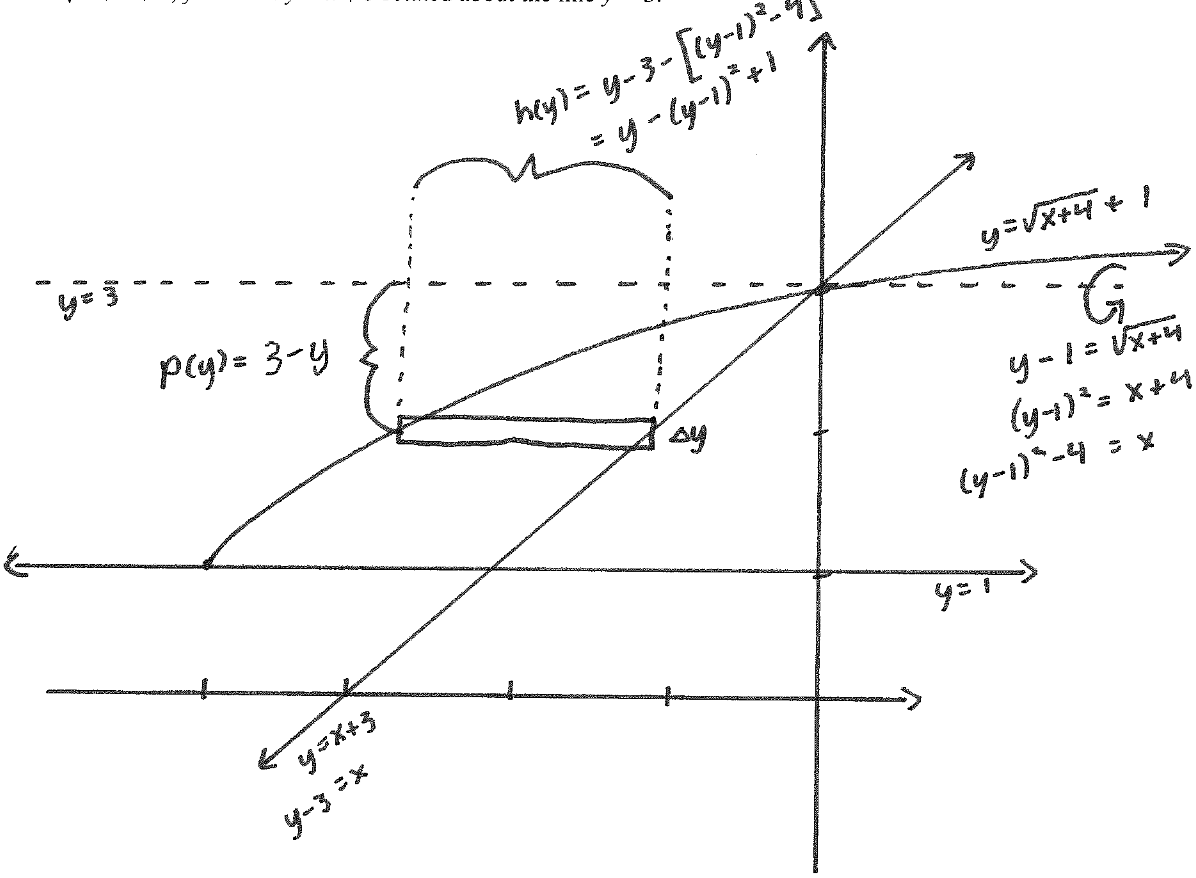
$$= \left[\frac{-1}{x} - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1} + \left[\frac{x^2}{2} + \frac{1}{x}\right]_{1}^{2}$$

$$= \left[\frac{-1}{1} - \frac{1}{2}\right] - \left[\frac{-1}{\frac{1}{2}} - \frac{(\frac{1}{2})^2}{2}\right] + \left[\frac{2^2}{2} + \frac{1}{2}\right] - \left[\frac{1}{2} + \frac{1}{1}\right]$$

$$= \left[-\frac{3}{2}\right] - \left[-2 - \frac{1}{8}\right] + \left[2 + \frac{1}{2}\right] - \frac{3}{2}$$

$$= \frac{13}{8}$$

Question 7. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = \sqrt{x+4}+1$, $y = 1$ and $y = x+3$ rotated about the line $y = 3$.

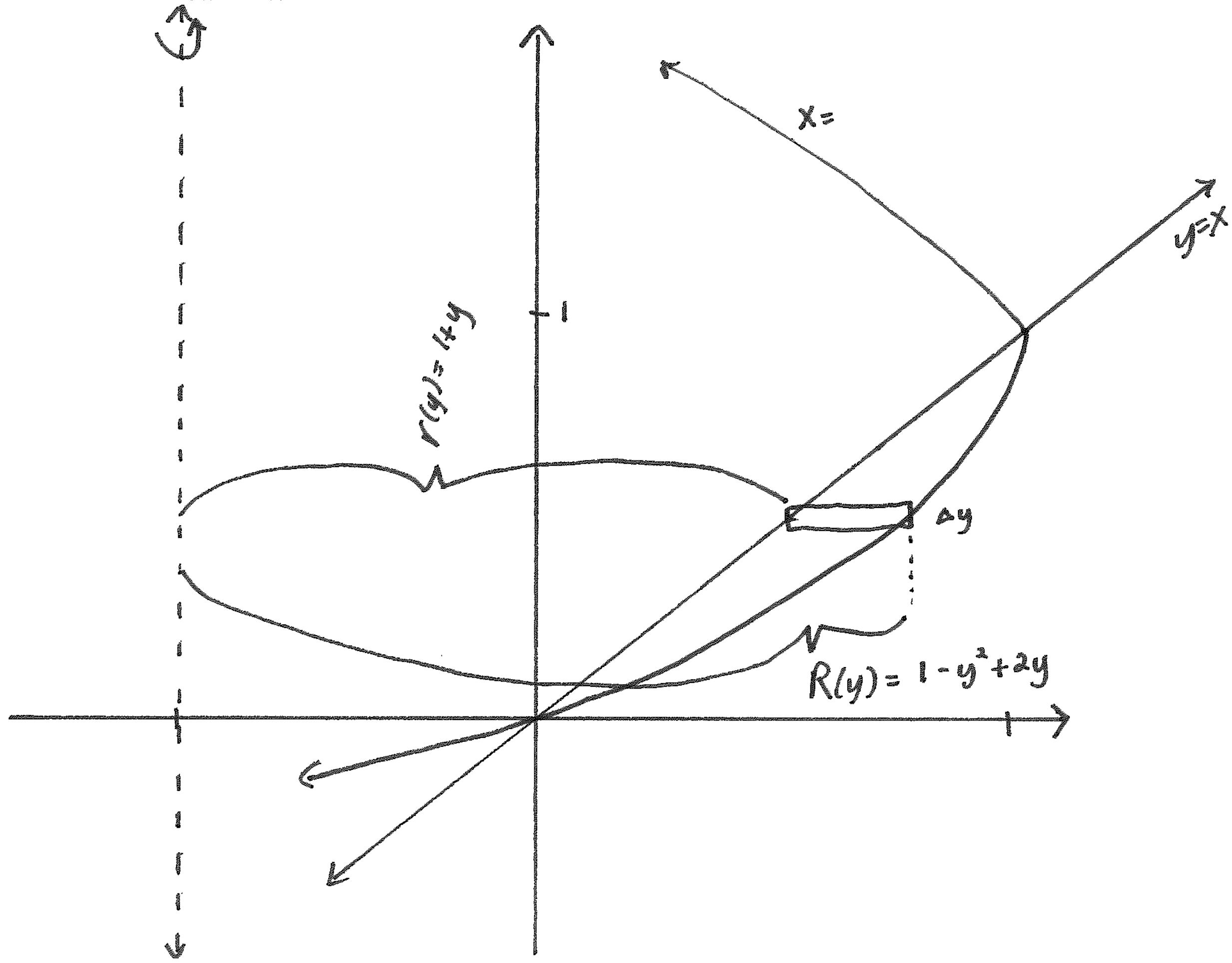


<u>rep. element:</u>

$$\Delta V = 2\pi p(y) h(y) \Delta y$$
$$= 2\pi (3-y)\left(y-(y-1)^2+1\right)\Delta y$$

$$V = \int_1^3 2\pi(3-y)\left(y-(y-1)^2+1\right)dy$$

Question 8. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $x = -y^2 + 2y$, $x = y$ rotated about the line $x = -1$.



Intersection of two curves:

$y = -y^2 + 2y$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0 \quad y = 1$

<u>rep. element:</u>

$\Delta V = \pi \left[(R(y))^2 - (r(y))^2 \right] \Delta y$

$= \pi \left[(1 + 2y - y^2)^2 - (1+y)^2 \right] \Delta y$

$V = \int_0^1 \pi \left[(1 + 2y - y^2)^2 - (1+y)^2 \right] dy$

Question 9. *(5 marks)* Find the length of the curve.

$$y = \ln(\cos x) \quad 0 \le x \le \frac{\pi}{3}$$

$$y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$$

$$S = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

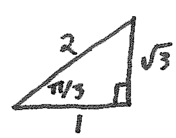
$$= \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx$$

$$\because \int_0^{\pi/3} \sec x\, dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/3}$$

$$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \underbrace{\ln|\overbrace{\sec 0}^{1} + \overbrace{\tan 0}^{0}|}_{0}$$

$$= \ln|2 + \sqrt{3}|$$



Bonus Question. *(3 marks)*
If $f(x)$ is a quartic function such that $f(0) = 4$, $f''(0) = 18$ and

$$\int \frac{f(x)}{x^2(x-1)^2(x^2+1)}\,dx$$

is a rational function, find the value of $f^{(4)}(x)$.

Since $f(x)$ is a quartic function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f(0) = 4 \Leftrightarrow e = 4$

$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f''(x) = 12ax^2 + 6bx + 2c$

$f''(0) = 18 \Leftrightarrow c = 9$

$$\frac{ax^4 + bx^3 + 9x^2 + dx + 4}{x^2(x-1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}$$

Since integral is rational $\Rightarrow A = 0,\ C = 0,\ E = 0,\ F = 0$

$ax^4 + bx^3 + 9x^2 + dx + 4 = B(x-1)^2(x^2+1) + Dx^2(x^2+1)$

$ax^4 + bx^3 + 9x^2 + dx + 4 = (B+D)x^4 - 2Bx^3 + Dx^2 - 2Bx + B$

$\therefore B = 4$ and $D = 9$

$\therefore a = B + D = 13$

and $f'''(x) = 24ax + 6b$, $f^{(4)}(x) = 24a$, $\therefore f^{(4)}(x) = 24(13)$