Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S1: Fall 2011

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Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\{\ln(3)-\ln(2), \ln(5)-\ln(3), \ln(7)-\ln(4), \ln(9)-\ln(5), \ldots\}$$

$$a_n = \ln(2n+1) - \ln(n+1)$$
$$= \ln\left(\frac{2n+1}{n+1}\right)$$

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \ln\left(\frac{2n+1}{n+1}\right)$$

$$= \ln(2)$$

$\therefore$ the sequence converges to $\ln(2)$

Question 2. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=3}^{\infty} \frac{-1}{(n-1)(n-2)}$$

$$\frac{-1}{(n-1)(n-2)} = \frac{A}{n-1} + \frac{B}{n-2}$$

$$-1 = A(n-2) + B(n-1)$$

Let $n=1$

$$-1 = A(1-2) + B(1-1)$$

$$1 = A$$

Let $n=2$

$$-1 = A(2-2) + B(2-1)$$

$$-1 = B$$

$$\therefore \sum_{n=3}^{\infty} \frac{-1}{(n-1)(n-2)} = \sum_{n=3}^{\infty}\left[\frac{1}{n-1} - \frac{1}{n-2}\right]$$

Lets look at the partial sum

$$S_n = a_3 + a_4 + a_5 + \dots + a_{n-2} + a_{n-1} + a_n$$

$$= \left[\frac{1}{\cancel{3-1}} - \frac{1}{3-2}\right] + \left[\frac{1}{\cancel{4-1}} - \frac{1}{\cancel{4-2}}\right] + \left[\frac{1}{5-1} - \frac{1}{\cancel{5-2}}\right] + \dots +$$

$$\left[\frac{1}{\cancel{n-2-1}} - \frac{1}{n-2-2}\right] + \left[\frac{1}{\cancel{n-1-1}} - \frac{1}{\cancel{n-1-2}}\right] + \left[\frac{1}{n-1} - \frac{1}{\cancel{n-2}}\right]$$

$$= \frac{-1}{1} + \frac{1}{n-1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} -1 + \cancel{\frac{1}{n-1}}^{\,0}$$

$$= -1$$

Question 3. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{2^{n+2} - 3^{n-1}}{4^{n+1}}$$

Lets compute the sum of $\sum_{n=1}^{\infty} \frac{2^{n+2}}{4^{n+1}}$ if it converges

converges since $|r| = \frac{1}{2} < 1$

$$= \sum_{n=1}^{\infty} \frac{2^{n+2}}{4^{n+1}} = \sum_{n=1}^{\infty} \frac{2^2 2^n}{4 4^n} = \sum_{n=0}^{\infty} \left(\frac{2}{4}\right)^n - a_0 = \frac{1}{1-\frac{1}{2}} - 1$$

$$= 2 - 1 = 1$$

Lets compute the sum of $\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+1}}$ if it converges

converges since $|r| = \frac{3}{4} < 1$

$$= \sum_{n=1}^{\infty} \frac{3^{-1} 3^n}{4 4^n} = \sum_{n=1}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^n = \sum_{n=0}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^n - a_0 = \frac{1/12}{1-\frac{3}{4}} - \frac{1}{12}$$

$$= \frac{1}{3} - \frac{1}{12}$$

$$= \frac{1}{4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^{n+2} - 3^{n-1}}{4^{n+1}} = \sum_{n=1}^{\infty} \frac{2^{n+2}}{4^{n+1}} - \sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+1}}$$

$$= 1 - \frac{1}{4}$$

$$= 3/4$$

Question 4. *(5 marks)* Determine whether the series is absolutely convergent, or conditionally convergent, or divergent.

$$\sum_{n=5}^{\infty} \frac{n!5^n(-1)^n}{(2n)!}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!\,5^{n+1}(-1)^{n+1}}{(2(n+1))!}}{\frac{n!\,5^n(-1)^n}{(2n)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!\,5^{n+1}(-1)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!\,5^n(-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}(n+1)\cancel{5^n}5\cancel{(2n)!}}{\cancel{(2n)!}(2n+1)(2n+2)\cancel{n!}\cancel{5^n}}$$

$$= \lim_{n\to\infty} \frac{5(n+1)}{(2n+1)(2n+2)}$$

$$= 0 < 1$$

$\therefore$ the series is absolutely convergent by the ratio test.

Question 5. *(5 marks)* Determine whether the series is absolutely convergent, or conditionally convergent, or divergent.

$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ Lets determine if the series is abs. conv.

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{n \ln n} \right| = \sum_{n=3}^{\infty} \frac{1}{n \ln n}$$

Lets apply the integral test

Let $f(x) = \frac{1}{x \ln x}$

- $f(x)$ is positive for $x \geq 2$.
- $f(x)$ is continuous for $x \geq 2$.
- Is $f(x)$ decreasing for $x \geq 2$? ✓

$$f'(x) = \frac{-1}{(x \ln x)^2} \left(\ln x + \frac{1}{x} x \right) < 0$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u} du$$

$u = \ln x$

$du = \frac{1}{x} dx$

$u(b) = \ln b$

$u(3) = \ln 2$

$$= \lim_{b \to \infty} \Big[\ln(u) \Big]_{\ln 2}^{\ln b}$$

$$= \lim_{b \to \infty} \ln \underbrace{\ln b}_{\to \infty} - \ln \ln 2$$

$\therefore$ does not converge by integral test

$\therefore$ not abs. conv.

Lets determine if the series is cond. conv.

Lets apply the Alternating series test.

(i) $b_{n+1} < b_n$ where $b_n = \frac{1}{n \ln n}$ is true since $f'(x) < 0$

(ii) $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n \ln n} = 0$

$\therefore$ converges by the alternating series test

$\therefore$ conditionally convergent.

Question 6. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \underbrace{\frac{\sin^2 2n}{2^n}}_{a_n}$$

Note that the series is positive.

$$a_n = \frac{\sin^2 2n}{2^n} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n = b_n$$

$\sum b_n$ is convergent since geometric series where $|r| = \frac{1}{2} < 1$

$\therefore \sum a_n$ is convergent by comparison test since $\sum b_n$ is convergent and $a_n \le b_n$

Question 7. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=20}^{\infty} \underbrace{\left(\frac{1-2n+n^2}{n^2+9n+1001}\right)^{\pi}}_{a_n}$$

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \left(\frac{1-2n+n^2}{n^2+9n+1001}\right)^{\pi}$$

$$= 1^{\pi}$$

$$= 1 \neq 0$$

$\therefore$ diverges by the n^{th} term divergence test

Question 8. *(5 marks)* Find the radius of convergence and interval of convergence of the series.

$$\underbrace{\sum_{n=2}^{\infty}(-1)^n\frac{(x+1)^n}{n4^n\ln n}}_{a_n(x)}$$

Lets find the radius of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x+1)^{n+1}}{(n+1)4^{n+1}\ln(n+1)}}{\frac{(-1)^n(x+1)^n}{n4^n\ln n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)4^{n+1}\ln(n+1)}\cdot\frac{n4^n\ln n}{(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+1)}{4}\cdot\frac{\ln n}{\ln(n+1)}\cdot\frac{n}{n+1}\right|$$

$$= \frac{|x+1|}{4}\lim_{x\to\infty}\frac{\ln x}{\ln(x+1)}$$

$$= \frac{|x+1|}{4}\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}}$$

$$= \frac{|x+1|}{4}\lim_{x\to\infty}\frac{x+1}{x} = \frac{|x+1|}{4}$$

So

$$\frac{|x+1|}{4} < 1$$

$$|x+1| < 4 = R$$

$\therefore$ $R = 4$

and the series converges for value

$$-4 < x+1 < 4$$

$$-5 < x < 3$$

Lets verify the end point for convergence

Let $x = 3$

$$\sum_{n=2}^{\infty}\frac{(-1)^n(3+1)^n}{n4^n\ln n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$$ converges by Q.5

Let $x=-5$

$$\sum_{n=2}\frac{(-1)^n(-5+1)^n}{n4^n\ln n} = \sum_{n=2}^{\infty}\frac{(-1)^n(-1)^n4^n}{n4^n\ln n} = \sum_{n=2}^{\infty}\frac{1}{n\ln n}$$ diverges by Q.5

$\therefore$ the interval of convergence is $(-5, 3]$

Question 9. *(5 marks)* Find the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $x = 9$. Assume that f has a power series expansion. Do not show that $R_n \to 0$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$f(x) = x^{-1/2} \qquad f(9) = 9^{-1/2} = 3^{-1}$$

$$f'(x) = \frac{-1}{2} x^{-3/2} \qquad f'(9) = \frac{-1}{2} 9^{-3/2} = -\frac{1}{2} 3^{-3}$$

$$f''(x) = -\frac{1}{2}\left(-\frac{3}{2}\right) x^{-5/2} \qquad f''(9) = \frac{1}{2}\left(\frac{3}{2}\right)(9)^{-5/2} = \frac{1}{2}\left(\frac{3}{2}\right) 3^{-5}$$

$$f'''(x) = \frac{-1}{2}\left(-\frac{3}{2}\right)\left(\frac{-5}{2}\right) x^{-7/2} \qquad f'''(9) = -\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) 9^{-7/2} = \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) 3^{-7}$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)\ x^{\frac{-(2n+1)}{2}}}{2^n} \qquad f^{(n)}(9) = \frac{(-1)^n 1 \cdot 2 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n} 3^{-(2n+1)}$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 2 \cdot 3 \cdot 5 \ldots (2n-1)\ 3^{-(2n+1)}}{2^n} \cdot \frac{(x-9)^n}{n!} + \frac{1}{3}$$

Bonus Question. *(3 marks)* Prove: If $\sum a_n$ is convergent then $\sum 1/a_n$ is divergent.

If $\sum a_n$ is convergent we have shown in class that $\lim_{n\to\infty} a_n = 0$

Let $\sum b_n$ where $b_n = \frac{1}{a_n}$, lets apply the n^{th} term divergence test

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\cancel{a_n}} \to \infty \quad \text{since} \quad \lim_{n\to\infty} a_n = 0$$

$\therefore \sum b_n$ diverges by the n^{th} term divergence test.