Dawson College: Calculus II (REGULAR): 201-NYB-05-S4: Fall 2011

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Test 1

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate using the definition of the definite integral

$$\int_1^3 x^2-4x-2\,dx. = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 1 + \frac{2i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left[\left(1+\frac{2i}{n}\right)^2 - 4\left(1+\frac{2i}{n}\right) - 2\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left[1+\frac{4i}{n}+\frac{4i^2}{n^2}-4-\frac{8i}{n}-2\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i^2}{n^2}-\frac{4i}{n}-5\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\sum_{i=1}^{n}\frac{4i^2}{n^2} - \sum_{i=1}^{n}\frac{4i}{n} - \sum_{i=1}^{n}5\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{4}{n}\sum_{i=1}^{n} i - 5n\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{\cancel{4}^{\,2}}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,3} - \frac{\cancel{4}^{\,2}}{\cancel{n}}\,\frac{\cancel{n}(n+1)}{\cancel{2}} - 5n\right]$$

$$= \lim_{n\to\infty}\left[\frac{4(n+1)(2n+1)}{3n^2} - \frac{4(n+1)}{n} - \frac{10n}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{4(n+1)}{3n}\,\frac{(2n+1)}{n} - \frac{4(n+1)}{n} - 10\right]$$

$$= \frac{4}{3}\cdot 2 - 4 - 10$$

$$= \frac{8}{3} - 14 = -\frac{34}{3}$$

Question 2. *(1 mark each)* Evaluate the following:

a.

$$\int_{\sqrt{2}}^{\sqrt{2}} x^2+\frac{1}{1+x^2}\,dx. = 0 \qquad \text{since } \int_a^a f(x)\,dx = 0$$

b.

$$\int \sec x\,dx = \ln|\sec x+\tan x| + c$$

c.

$$\int \tan x\,dx = -\ln|\cos x| + c$$

d.

$$\int \frac{1}{\sqrt[4]{x^3}}\,dx = \int x^{-3/4}\,dx = 4x^{1/4} + C$$

e. True or False: If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b kf(k)\,dk = k\int_a^b f(k)\,dk$$

False, k is not a constant.

Question 3. *(5 marks)* Evaluate the definite integral:

$$\int_{\pi/4}^{\pi/3} \frac{\sin\theta+\sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sin\theta\,(1+\tan^2\theta)\,d\theta}{\sec^2\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sin\theta\,\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \sin\theta\,d\theta$$

$$= \Big[-\cos\theta\Big]_{\pi/4}^{\pi/3}$$

$$= -\cos\frac{\pi}{3} + \cos\frac{\pi}{4}$$

$$= -\frac{1}{2} + \frac{1}{\sqrt{2}}$$

$$= \frac{2-\sqrt{2}}{2\sqrt{2}}$$

uestion 4. *(5 marks)* Evaluate the definite integral:

$$\int_1^2 \frac{z^2+1}{\sqrt[3]{z}}\,dz = \int_1^2 (z^2+1)\,z^{-1/3}\,dz = 3\cdot 2^{2/3} - \frac{15}{8}$$

$$= \int_1^2 z^{5/3} + z^{-1/3}\,dz$$

$$= \left[\frac{3z^{8/3}}{8} + \frac{3z^{2/3}}{2}\right]_1^2$$

$$= \left[\frac{3\,2^{8/3}}{8} + \frac{3\,2^{2/3}}{2}\right] - \left[\frac{3\,1^{8/3}}{8} + \frac{3\,1^{2/3}}{2}\right]$$

$$= \left[\frac{3\cdot \cancel{4}\,2^{2/3}}{\cancel{8}\,2} + \frac{3\cdot 2^{2/3}}{2}\right] - \left[\frac{15}{8}\right]$$

)uestion 5. *(5 marks)* Find the average value of the function

$$f(x) = \frac{1}{x\sqrt{x^2-4}}$$

n the interval $[2\sqrt{2},4]$.

$$f_{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{4-2\sqrt{2}}\int_{2\sqrt{2}}^{4} \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$= \frac{1}{4-2\sqrt{2}}\left[\frac{1}{2}\operatorname{arcsec}\frac{x}{2}\right]_{2\sqrt{2}}^{4}$$

$$= \frac{1}{4-2\sqrt{2}}\left[\left[\frac{1}{2}\operatorname{arcsec}\frac{4}{2}\right] - \left[\frac{1}{2}\operatorname{arcsec}\frac{2\sqrt{2}}{2}\right]\right]$$

$$= \frac{1}{4-2\sqrt{2}}\left[\frac{1}{2}\operatorname{arcsec}2 - \frac{1}{2}\operatorname{arcsec}\sqrt{2}\right]$$

$$= \frac{1}{4-2\sqrt{2}}\left[\frac{1}{2}\cdot\frac{\pi}{3} - \frac{1}{2}\cdot\frac{\pi}{4}\right]$$

$$= \frac{1}{2(4-2\sqrt{2})}\cdot\frac{\pi}{12}$$

$$= \frac{\pi}{24(4-2\sqrt{2})}$$

Question 6. *(5 marks)* Evaluate the expression:

$\frac{d}{dx}\left[\int_{2x}^{\cot 3x} (\cos t)^t \, dt\right]$ Let $h(x) = \int_{2x}^{\cot 3x} (\cos t)^t \, dt$

where $f(x) = \int_0^x (\cos t)^t dt$

$$= \int_{2x}^{0} (\cos t)^t dt + \int_0^{\cot 3x} (\cos t)^t dt$$

and $f'(x) = (\cos x)^x$ by 2^{nd} FTC

$g_1(x) = 2x \Rightarrow g_1'(x) = 2$

$g_2(x) = \cot 3x \Rightarrow g_2'(x) = -3\csc^2(3x)$

$$= -\int_0^{2x} (\cos t)^t dt + \int_0^{\cot 3x} (\cos t)^t dt$$

$$= -f(g_1(x)) + f(g_2(x))$$

So $h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$

$$= -f'(2x)\cdot 2 + f'(\cot 3x)(-3\csc^2(3x))$$

$$= -2(\cos(2x))^{2x} - 3\csc^2(3x)(\cos(\cot 3x))^{\cot 3x}$$

Question 7. *(5 marks)* If $\int_1^{\sqrt{3}} f(x)\,dx = 22$ then evaluate

$\int_{\sqrt{3}}^{1} 24x^3 + 3f(x)\,dx = \int_{\sqrt{3}}^{1} 24x^3 dx + \int_{\sqrt{3}}^{1} 3f(x)dx$

$$= \left[\frac{24x^4}{4}\right]_{\sqrt{3}}^{1} - \int_1^{\sqrt{3}} 3f(x)dx$$

$$= \left[\frac{24\cdot 1^4}{4} - \frac{24(\sqrt{3})^4}{4}\right] - 3\int_1^{\sqrt{3}} f(x)dx$$

$$= 6 - 6\cdot 9 - 3\cdot 22$$

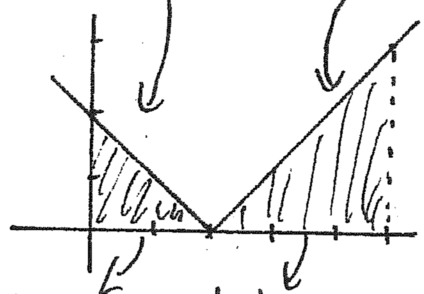
$$= 6 - 54 - 66$$

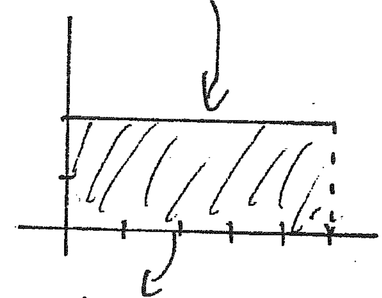
$$= -114$$

Question 8. *(5 marks)* Evaluate the integral by interpreting it in terms of areas.

$$\int_0^5 |x-2|+2\,dx = \int_0^5 |x-2|\,dx + \int_0^5 2\,dx$$

$$= \int_0^2 |x-2|\,dx + \int_2^5 |x-2|\,dx + \int_0^5 2\,dx$$

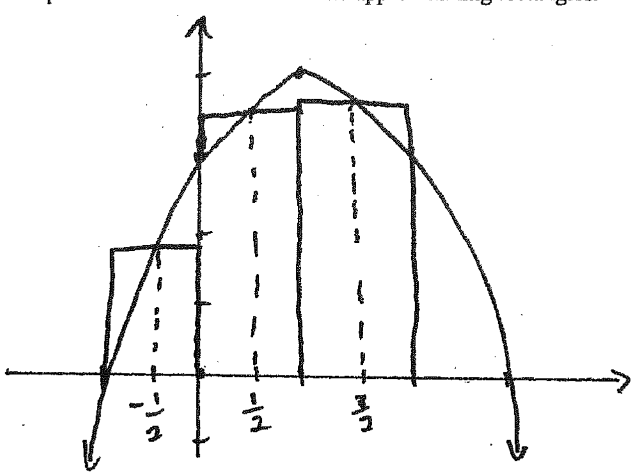




$$= \frac{b_1h_1}{2} + \frac{b_2h_2}{2} + lw$$

$$= \frac{2(2)}{2} + \frac{3(3)}{2} + 2(5) = \frac{33}{2}$$

Question 9. *(5 marks)* Estimate the area under the graph of $f(x) = 4-(x-1)^2$ from $x=-1$ to $x=2$ using three rectangles and using the midpoints. Sketch the curve and the approximating rectangles.



$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{3} = \frac{3}{3} = 1$$

$$x_i = a + i\Delta x = -1 + i$$

$x_0 = -1$, $x_1 = 0$: $x_1^* = \frac{x_0+x_1}{2} = \frac{-1+0}{2} = -\frac{1}{2}$

$x_1 = 0$, $x_2 = 1$: $x_2^* = \frac{x_1+x_2}{2} = \frac{0+1}{2} = \frac{1}{2}$

$x_2 = 1$, $x_3 = 2$: $x_3^* = \frac{x_2+x_3}{2} = \frac{1+2}{2} = \frac{3}{2}$

$$\text{Area} \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x$$

$$= \left[4-\left(-\tfrac{1}{2}-1\right)^2\right] + \left[4-\left(\tfrac{1}{2}-1\right)^2\right] + \left[4-\left(\tfrac{3}{2}-1\right)^2\right]$$

$$= \left[4-\tfrac{9}{4}\right] + \left[4-\tfrac{1}{4}\right] + \left[4-\tfrac{1}{4}\right]$$

$$= \frac{37}{4}$$

Question 10. *(5 marks)* Find the value(s) of c such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

for $f(x) = (x-3)^2$ on the interval $[2,5]$. Sketch the curve, indicate the c value(s).

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$(c-3)^2 = \frac{1}{5-2}\int_2^5 (x-3)^2 dx$$

$$(c-3)^2 = \frac{1}{3}\int_2^5 x^2-6x+9\, dx$$

$$(c-3)^2 = \frac{1}{3}\left[\frac{x^3}{3} - 3x^2 + 9x\right]_2^5$$

$$(c-3)^2 = \frac{1}{3}\left[\left[\frac{5^3}{3} - 3\cdot5^2 + 9\cdot5\right] - \left[\frac{2^3}{3} - 3\cdot2^2 + 9(2)\right]\right]$$

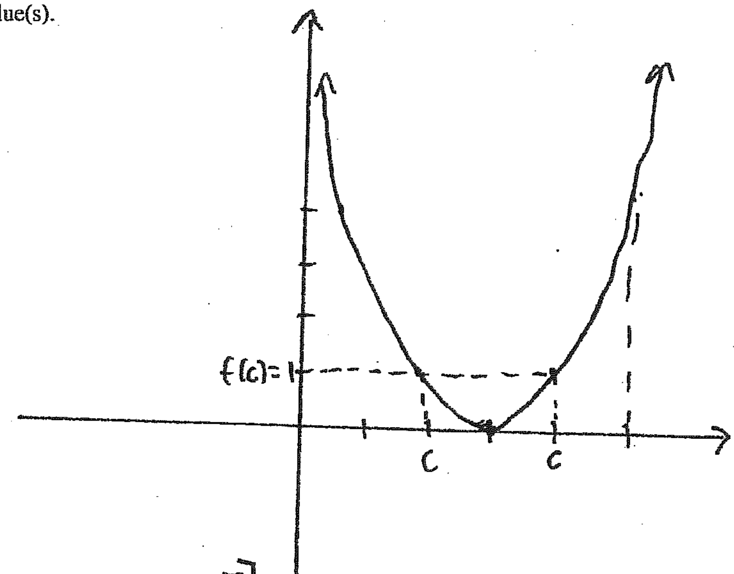
$$(c-3)^2 = \frac{1}{3}\left[\frac{125}{3} - 75 + 45\right] - \left[\frac{8}{3} - 12 + 18\right]$$

$$(c-3)^2 = 1$$

$$c-3 = \pm 1$$

$$c = \pm 1 + 3$$

$$c = 4 \text{ or } c = 2$$



Bonus Question. *(3 marks)*
Prove: If f and $g(x)$ is a continuous on [a,b] and c is a constant, then

$$\int_a^b f(x) + cg(x)\,dx = \int_a^b f(x)\,dx + c\int_a^b g(x)\,dx$$

$$LHS = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i) + cg(x_i)]\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i)\Delta x + cg(x_i)\Delta x]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} cg(x_i)\Delta x\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty}\sum_{i=1}^{n} cg(x_i)\Delta x$$

$$= \int_a^b f(x)dx + c\lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \int_a^b f(x)dx + c\int_a^b g(x)\,dx$$