Dawson College: Calculus II (REGULAR): 201-NYB-05-S4: Fall 2011

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Test 3

This test is graded out of 43 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Find the limit.

$$\lim_{x\to 1} \frac{3e^{1-x} - \ln x + \sin(4x-4) - 3}{x^2 - 2x + 1}$$

I.f. $\frac{3-0+0-3}{1-2+1} = \frac{0}{0}$

$$\overset{\hat{H}}{=} \lim_{x\to 1} \frac{-3e^{1-x} - \frac{1}{x} + \cos(4x-4)\cdot 4}{2x-2}$$

I.f. $\frac{-3-1+4}{2-2} = \frac{0}{0}$

$$\overset{\hat{H}}{=} \lim_{x\to 1} \frac{3e^{1-x} + \frac{1}{x^2} - \sin(4x-4)\cdot 16}{2}$$

$$= \frac{3+1-0}{2} = 4$$

Question 2. *(5 marks)* Find the limit.

$$y = \lim_{x\to 0^+} (\cos x)^{1/x^2}$$

$$\ln y = \ln \lim_{x\to 0^+} (\cos x)^{1/x^2}$$

$$\ln y = \lim_{x\to 0^+} \ln (\cos x)^{1/x^2}$$

$$\ln y = \lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2} \quad \text{I.f. } \frac{0}{0}$$

$$\ln y = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}\,(-\sin x)}{2x} \quad \text{by } \hat{H}$$

$$\ln y = \lim_{x\to 0^+} \frac{-\tan x}{2x} \quad \text{I.f. } \frac{0}{0}$$

$$\ln y = \lim_{x\to 0^+} \frac{-\sec^2 x}{2} \quad \text{by } \hat{H}$$

$$\ln y = \frac{-1}{2}$$

$$y = e^{-1/2}$$

Question 3. *(1 mark each)*

a. *(1 mark)* True or False: Suppose that $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx \qquad \text{False}$$

b. *(2 marks)* True or False, Justify.

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\cos x}{x}$$

False, in order to use $\hat{H}$ the I.F. must be $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Question 4. *(5 marks)* Evaluate the improper integral or show it diverges:

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{x^2}{9+(x^3)^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{x^2}{9+(x^3)^2}\,dx$$

$u = x^3$
$du = 3x^2\,dx$
$\frac{du}{3} = x^2\,dx$
$u(0) = 0$
$u(a) = a^3$
$u(b) = b^3$

$$= \lim_{a\to-\infty} \int_{a^3}^{0} \frac{1}{9+u^2}\frac{du}{3} + \lim_{b\to\infty} \int_{0}^{b^3} \frac{1}{9+u^2}\frac{du}{3}$$

$$= \lim_{a\to-\infty} \frac{1}{3}\left[\frac{1}{3}\arctan\frac{u}{3}\right]_{a^3}^{0}$$

$$+ \lim_{b\to\infty} \frac{1}{3}\left[\frac{1}{3}\arctan\frac{u}{3}\right]_{0}^{b^3}$$

$$= \lim_{a\to-\infty} \frac{1}{3}\left[\underbrace{\frac{1}{3}\arctan\frac{0}{3}}_{0} - \frac{1}{3}\arctan\frac{a^3}{3}\right] \quad \arctan\frac{a^3}{3} \to -\pi/2$$

$$+ \lim_{b\to\infty} \frac{1}{3}\left[\frac{1}{3}\arctan\frac{b^3}{3} - \underbrace{\frac{1}{3}\arctan\frac{0}{3}}_{0}\right] \quad \arctan\frac{b^3}{3} \to \pi/2$$

$$= \frac{\pi}{18} + \frac{\pi}{18}$$

$$= \frac{\pi}{9}$$

Question 5. *(5 marks)* Evaluate the improper integral or show it diverges:

$\int_0^2 z^2 \ln z \, dz$ infinite discontinuity at $x=0$

$$= \lim_{a\to 0^+} \int_a^2 z^2 \ln z \, dz \qquad u = \ln z \quad du = \frac{1}{z}dz \qquad v = \frac{z^3}{3} \quad dv = z^2 dz$$

$$= \lim_{a\to 0^+} \left[[uv]_a^2 - \int_a^2 v\,du \right]$$

$$= \lim_{a\to 0^+} \left[\left[\frac{z^3}{3}\ln z\right]_a^2 - \int_a^2 \frac{z^{\cancel{3}\,2}}{3}\frac{1}{\cancel{z}}dz \right]$$

$$= \lim_{a\to 0^+} \left[\frac{2^3}{3}\ln 2 - \frac{a^3}{3}\ln a - \left[\frac{z^3}{9}\right]_a^2 \right]$$

I.F. $0\cdot -\infty$

$$= \lim_{a\to 0^+} \left[\frac{8}{3}\ln 2 - \frac{\ln a}{3a^{-3}} - \frac{2^3}{9} + \frac{\cancel{a^3}}{9} \right] \quad \nearrow 0$$

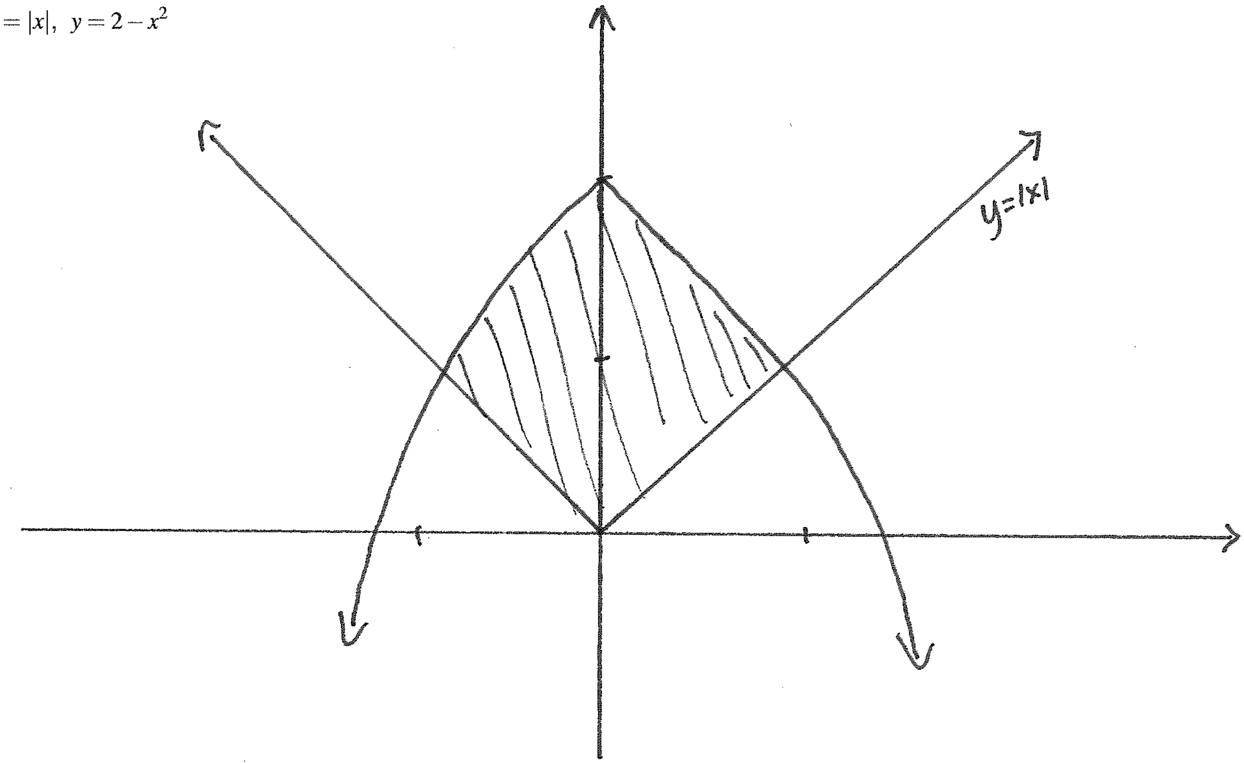
I.F $\frac{-\infty}{\infty}$

$$= \frac{8}{3}\ln 2 - \frac{8}{9} - \lim_{a\to 0^+} \frac{1/a}{-9a^{-4}} \text{ by } \hat{H}$$

$$= \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9}\lim_{a\to 0^+} \frac{a^{\cancel{4}\,3}}{\cancel{a}} = \frac{8}{3}\ln 2 - \frac{8}{9}$$

Question 6. *(5 marks)* Sketch the region enclosed by the given curves. Then find the area of the region.

$y = |x|, \ y = 2 - x^2$



Lets find the intersection of $y=|x|$, $y=2-x^2$

If $x>0$ then $y=|x|=x$, so $x=2-x^2$

$$x^2+x-2=0$$
$$(x+2)(x-1)=0$$
$$x=1$$

If $x<0$ then $y=|x|=-x$, so $-x=2-x^2$

$$x^2-x-2=0$$
$$(x-2)(x+1)=0$$
$$x=-1$$

$$\text{Area} = \int_{-1}^{0} 2-x^2-(-x)\,dx + \int_0^1 2-x^2-x\,dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} + \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

$$= -\left[2(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right] + \left[2(1) - \frac{1^3}{3} - \frac{1^2}{2}\right]$$

$$= \frac{7}{3}$$

Question 7. *(5 marks)* Find the arc length of the graph of the function

$$x = \frac{1}{3}\sqrt{y}(y-3) = \frac{1}{3}y^{3/2} - y^{1/2}$$

$$y' = \frac{1}{2}y^{1/2} - \frac{1}{2y^{1/2}}$$

over the interval $[1,9]$.

$$S = \int_a^b \sqrt{1 + (x')^2}\, dy$$

$$= \int_1^9 \sqrt{1 + \left(\frac{y^{1/2}}{2} - \frac{1}{2y^{1/2}}\right)^2}\, dy$$

$$= \int_1^9 \sqrt{1 + \frac{y}{4} - \frac{1}{2} + \frac{1}{4y}}\, dy$$

$$= \int_1^9 \sqrt{\frac{y}{4} + \frac{1}{2} + \frac{1}{4y}}\, dy$$

$$= \int_1^9 \sqrt{\left(\frac{y^{1/2}}{2} + \frac{1}{2y^{1/2}}\right)^2}\, dy$$

$$= \int_1^9 \frac{y^{1/2}}{2} + \frac{1}{2y^{1/2}}\, dy$$

$$= \left[\frac{2y^{3/2}}{6} + y^{1/2}\right]_1^9$$

$$= \frac{2\,9^{3/2}}{6} + 9^{1/2} - \left[\frac{2(1)^{3/2}}{6} + 1^{1/2}\right]$$

$$= 9 + 3 - \frac{1}{3} - 1$$

$$= \frac{32}{3}$$

Question 8. *(5 marks)* Find the volume of the solid obtained when the region bounded by the graphs of $y = 4x - x^2$, $y = x$ is rotated about the line x-axis.

Intersection of two curves: $4x - x^2 = x$

$0 = 3x - x^2$

$0 = x(3 - x)$

$x = 0 \quad x = 3$

$y = 4x - x^2$

y-int: $(0,0)$

x-int: $0 = 4x - x^2$

$0 = x(4-x)$

$x = 0 \quad x = 4$

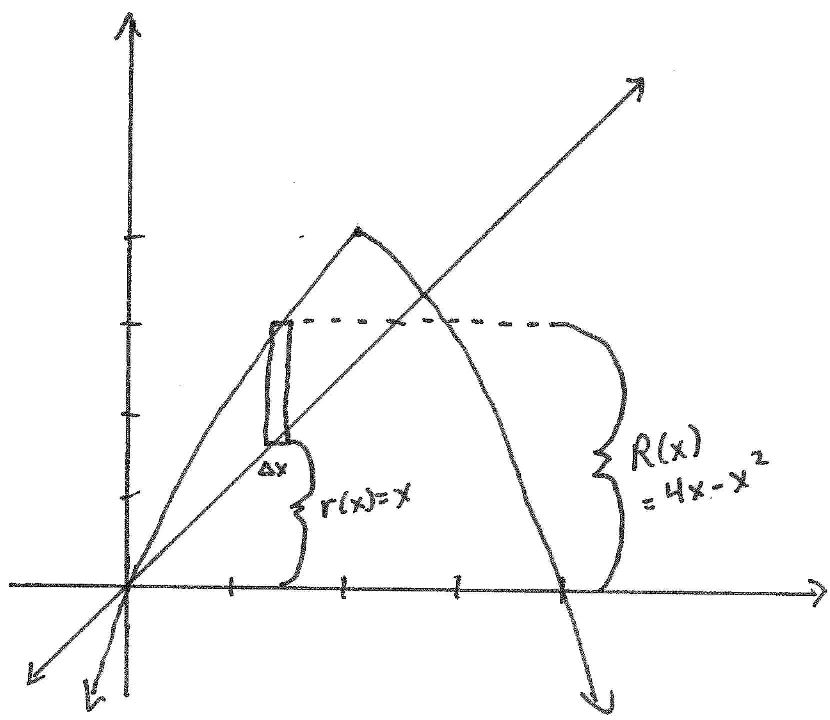
vertex:

$y = -[x^2 - 4x + 4 - 4]$

$= -[(x-2)^2 - 4]$

$= -(x-2)^2 + 4$

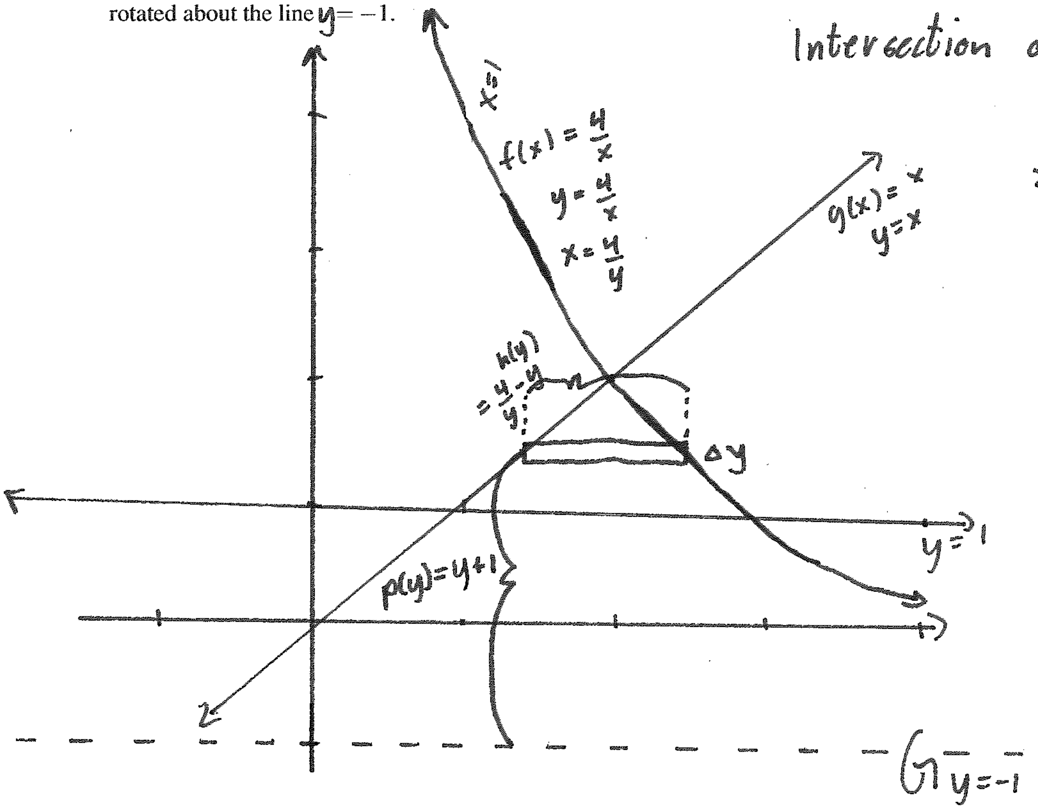
$V(2,4)$



$$\Delta V = \pi\left[(R(x))^2 - (r(x))^2\right]\Delta x$$
$$= \pi\left[(4x - x^2)^2 - x^2\right]\Delta x$$
$$= \pi\left[16x^2 - 8x^3 + x^4 - x^2\right]\Delta x$$

$$V = \pi\int_0^3 16x^2 - 8x^3 + x^4 - x^2\,dx$$
$$= \pi\left[\frac{16x^3}{3} - 2x^4 + \frac{x^5}{5} - \frac{x^3}{3}\right]_0^3$$
$$= \pi\left[\frac{16\,3^3}{3} - 2\,3^4 + \frac{3^5}{5} - \frac{3^3}{3}\right]$$
$$= \frac{108\pi}{5}$$

Question 9. *(5 marks)* Find the volume of the solid obtained when the region bounded by the graphs of $f(x) = \frac{4}{x}$, $y=1$ and $g(x) = x$ is rotated about the line $y = -1$.



Intersection of $f(x)$ and $g(x)$

$x = \frac{4}{x}$

$x^2 = 4$

$x = \pm 2$

$\Delta V = 2\pi\, p(y) h(y)\, \Delta y$

$\Delta V = 2\pi (y+1)\left(\frac{4}{y} - y\right)\Delta y$

$$V = \int_1^2 2\pi (y+1)\left(\frac{4}{y} - y\right) dy$$

$$= 2\pi \int_1^2 4 - y^2 + \frac{4}{y} - y \, dy$$

$$= 2\pi \left[4y - \frac{y^3}{3} + 4\ln|y| - \frac{y^2}{2}\right]_1^2$$

$$= 2\pi \left[\left[4(2) - \frac{2^3}{3} + 4\ln 2 - \frac{2^2}{2}\right] - \left[4(1) - \frac{1^3}{3} + 4\ln 1 - \frac{1}{2}\right]\right]$$

$$= 2\pi\left[8 - \frac{8}{3} + 4\ln 2 - 2 - 4 + \frac{1}{3} + \frac{1}{2}\right]$$

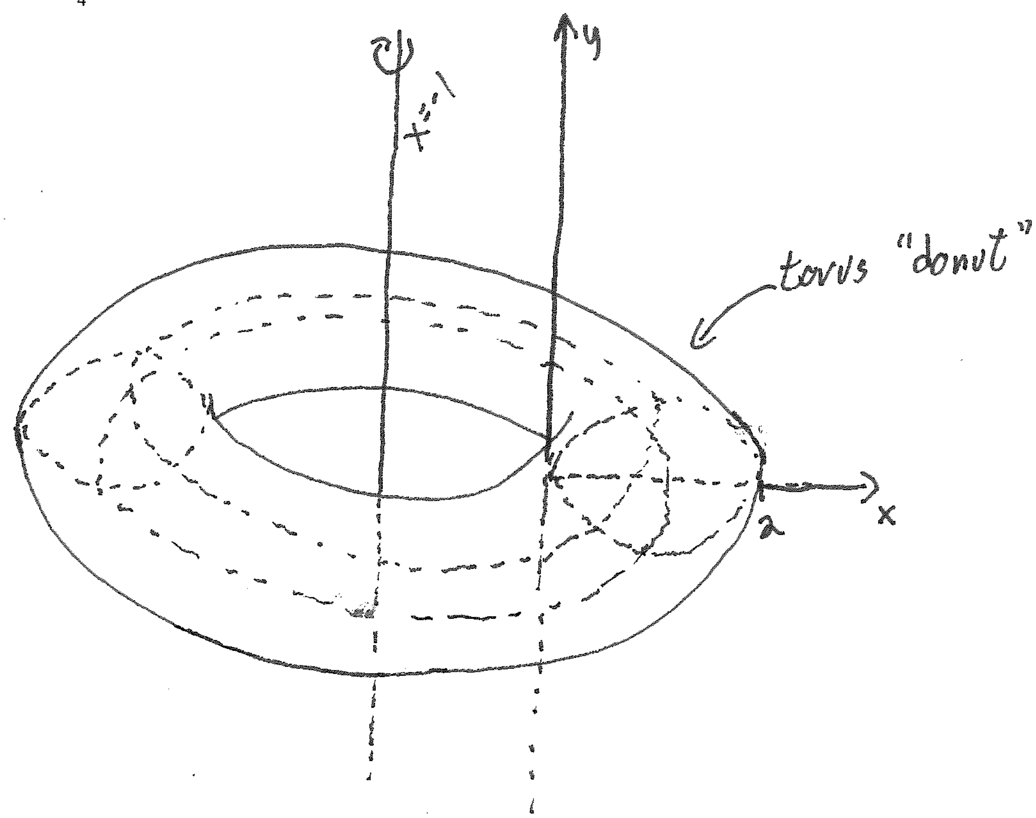
$$= \frac{\pi}{3} + 8\pi \ln 2$$

Bonus Question. *(1.5 marks each)*
Sketch and label distances of the solid obtained when the region bounded by the graphs of:

a. $(x-1)^2+y^2=1$ is rotated about the line $x=-1$.

b. $x^2+\frac{(y-2)^2}{4}=1$ is rotated about the line x-axis.

a)



b)