ALGEBRA MODULES (Revised Aug. 2000)

ALGEBRA MODULE ONE

POLYNOMIALS, DIVISION AND FACTORIZATION

LONG DIVISION OF POLYNOMIALS

Example 1: Divide the polynomial $f(x) = x^3 + 4x^2 - x - 4$ by $x - 1$ and use the result to factor $f(x)$ completely.

$$
\begin{array}{rll}
 & x^2 + 5x + 4 & \\
x - 1 \big) & x^3 + 4x^2 - x - 4 & \\
 & x^3 - x^2 & \text{Multiply: } x^2(x-1) \\
\hline
 & 5x^2 - x & \text{Subtract} \\
 & 5x^2 - 5x & \text{Multiply: } 5x(x-1) \\
\hline
 & 4x - 4 & \text{Subtract} \\
 & 4x - 4 & \text{Multiply } 4(x-1) \\
\hline
 & 0 & \text{Subtract}
\end{array}
$$

We see that $x^3 + 4x^2 - x - 4 = (x-1)(x^2 + 5x + 4)$ and by factoring the quadratic, we have $f(x) = (x-1)(x+1)(x+4)$.

Example 2: Divide $x^3 - 8$ by $x - 2$.
Since there are powers of x missing in $x^3 - 8$, we can choose to write those powers with zero coefficients or to leave spaces.

$$
\begin{array}{rl}
 & x^2 + 2x + 4 \\
x - 2 \big) & x^3 + 0x^2 + 0x - 8 \\
 & x^3 - 2x^2 \\
\hline
 & 2x^2 \\
 & 2x^2 - 4x \\
\hline
 & 4x - 8 \\
 & 4x - 8 \\
\hline
 & 0
\end{array}
$$

Therefore $x^3 - 8 = (x-2)(x^2 + 2x + 4)$

THE REMAINDER THEOREM

Suppose that $p(x)$ is a polynomial and r is a real number. If $p(x)$ is divided by $x - r$, the remainder is $p(r)$.

From the two examples above, we can see that if $x - r$ is a factor of a polynomial then the remainder after division is zero. The Remainder Theorem can be used as a test to see whether an expression is a factor of a polynomial.

Example: Use long division to find the remainder when $p(x) = 2x^3 - 2x^2 + 3x - 5$ is divided by $x - 2$.

$$
\begin{array}{r}
2x^2 + 2x + 7 \\
x - 2 \overline{\smash{\big)}\, 2x^3 - 2x^2 + 3x - 5} \\
\underline{2x^3 - 4x^2 \qquad\qquad} \\
2x^2 + 3x \qquad \\
\underline{2x^2 - 4x \qquad} \\
7x - 5 \\
\underline{7x - 14} \\
9
\end{array}
$$

The remainder after division is 9. Note that $p(2) = 2(2)^3 - 2(2)^2 + 3(2) - 5 = 16 - 8 + 6 - 5 = 9$. Obviously $x - 2$ is not a factor of $p(x)$. The result may be written as a polynomial plus a fraction:

$$\frac{2x^3 - 2x^2 + 3x - 5}{x - 2} = 2x^2 + 2x + 7 + \frac{9}{x - 2}.$$

The following theorem guarantees the condition under which an expression is a factor of a polynomial:

THE FACTOR THEOREM

If $p(x)$ is a polynomial and r a real number, then $p(x)$ has $x - r$ as a factor if and only if $p(r) = 0$.

Example: Show that $x - 3$ is a factor of $p(x) = x^3 - 2x^2 - 2x - 3$.

$p(3) = 3^3 - 2(3)^2 - 2(3) - 3 = 27 - 18 - 6 - 3 = 0$. Therefore by the Factor Theorem $x - 3$ is a factor of $p(x)$.

THE RATIONAL ZEROS TEST

If the polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots\dots\dots + a_2 x^2 + a_1 x + a_0$ has integer coefficients, then every rational zero, r, of $p(x)$ has the form $r = p/q$ where p and q have no common factors other than 1 and p is a factor of the constant term, a_0, while q is a factor of the leading coefficient, a_n.

This theorem is somewhat easier to apply when $a_n = 1$.

In the following two xamples find at least one rational zero of the polynomial and use the Remainder Theorem and long division to factor the polynomial completely.

Example 1: $p(x) = x^4 - x^3 + x^2 - 3x - 6$.

Possible zeros $= \pm 1, \pm 2, \pm 3, \pm 6$. When we check we find that the only two that give a remainder of zero are $x = -1$ and $x = 2$. If we divide $p(x)$ by $x + 1$ we find the quotient is $x^3 - 2x^2 + 3x - 6$. If we divide this quotient by $x - 2$, we obtain a quotient of $x^2 + 3$. So $p(x) = (x + 1)(x - 2)(x^2 + 3)$.

Example 2: $p(x) = x^3 - 3x^2 - x + 3$.
The possible rational zeros are the factors of 3 since the leading coefficient is 1. Possible zeros $= \pm 1, \pm 3$. It is easy to verify that $p(1) = 0$, so $x - 1$ is a factor. Using long division:

$$\begin{array}{r|l} & x^2 - 2x - 3 \\ \hline x-1 & x^3 - 3x^2 - x + 3 \\ & x^3 - x^2 \\ \hline & \quad -2x^2 - x \\ & \quad -2x^2 + 2x \\ \hline & \qquad\quad -3x + 3 \\ & \qquad\quad -3x + 3 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

$$p(x) = (x-1)(x-3)(x+1)$$

SPECIAL FACTORIZATIONS

DIFFERENCE OF SQUARES: $a^2 - b^2 = (a-b)(a+b)$
PERFECT SQUARES: $(a-b)^2 = a^2 - 2ab + b^2$ $\quad (a+b)^2 = a^2 + 2ab + b^2$
DIFFERENCE OF CUBES: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
SUM OF CUBES: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
CUBE OF A SUM: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
CUBE OF A DIFFERENCE: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
SECOND DEGREE POLYNOMIALS : Polynomials of the form $x^2 + (a+b)x + ab$ can be factored into $(x+a)(x+b)$; i.e., $x^2 + (a+b)x + ab = (x+a)(x+b)$

Example 1: $x^2 - 3x - 10 = (x-5)(x+2)$
Here $a = -5$ $\quad b = 2$ Note that $a + b = -5 + 2 = -3$ and $ab = (-5)(2) = -10$.

If the coefficient of x^2 is NOT 1 (one) we can factor the quadratic, in simple cases by TRIAL AND ERROR. Alternately we can use the GROUPING AND FACTORING method.

Example 2A: (TRIAL and ERROR).
FACTOR $\quad 3x^2 - 5x - 2$.
Solution: We need four terms a, b, c, d such that
$3x^2 - 5x - 2 = (a+b)(c+d)$ Thus $ac = 3x^2$, $bd = -2$.

i) Try $a = 3x$, $c = x$, $b = -1$, $d = 2$.
Thus $(3x-1)(x+2)$ which results in $3x^2 + 5x - 2$ and is INCORRECT.
ii) Try $a = 3x$ $\quad c = x$ $\quad b = 1$ $\quad d = -2$.
Thus $(3x+1)(x-2)$ which results in $3x^2 - 5x - 2$ which is CORRECT.
Thus $3x^2 - 5x - 2 = (3x+1)(x-2)$.

Example 2B: (GROUPING and FACTORING)

$3x^2 - 5x - 2$

$= 3x^2 - 6x + x - 2$ (SPLIT the middle term.)

$= 3x(x-2) + 1(x-2)$ (Grouping and Factoring)

$= (3x+1)(x-2)$

FOURTH DEGREE POLYNOMIALS: Fourth degree polynomials in the form $ax^4 + bx^2 + c$ may be factored as quadratics.

Example3: $x^4 - 5x^2 + 4 = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$

Example 4: $x^4 + 6x^2 - 7 = (x^2-1)(x^2+7) = (x-1)(x+1)(x^2+7)$

PROBLEMS

A. In the following problems, one of the zeros of the polynomial is given. Use the Factor Theorem and long division to factor the polynomial completely.

1) $x^3 - 4x^2 - x + 4;\ r = 1$

2) $x^3 + 5x^2 + 3x - 9;\ r = -3$

3) $x^3 - 3x^2 + 4;\ r = 2$

4) $x^4 - 9x^3 + 6x^2 + 16x;\ r = -1$

5) $x^3 - 3x^2 + 4x - 4;\ r = 2$

6) $2x^3 + x^2 - 11x - 10;\ r = -1$

B. Perform the indicated division and write the result as a polynomial plus a fraction:

1) $\dfrac{x^2+4x}{x^2-1}$

2) $\dfrac{x^3-2x^2+1}{x^2+1}$

3) $\dfrac{x^3+2}{x+2}$

4) $\dfrac{x^3+3x+5}{x+2}$

5) $\dfrac{x^4-2x^3-7x^2+4x+7}{x^3+2x^2+x}$

6) $\dfrac{x^4-3x^2+2}{x^3+4x}$

C. Factor the following polynomials completely using special factorizations or the rational zeros test.

1) $x^4 - 16$

2) $x^3 - 8$

3) $x^2 - 2x - 15$

4) $x^3 - 2x^2 - 24x$

5) $6x^2 + x - 2$

6) $8x^2 - 10x - 3$

7) $x^4 - 10x^2 + 9$

8) $8x^3 + 27$

9) $x^4 + x^2 - 2$

10) $x^3 - 2x^2 + 4x - 8$

11) $x^3 - 3x - 2$

12) $x^3 - 4x^2 + x + 6$

13) $x^3 - 5x^2 - 2x + 24$

14) $x^3 - 9x^2 + 20x - 12$

15) $x^4 + 3x^3 - 6x^2 - 28x - 24$

ANSWERS

A.

1) $(x-1)(x+1)(x-4)$

2) $(x+3)^2(x-1)$

3) $(x-2)^2(x+1)$

4) $x(x+1)(x-2)(x-8)$

5) $(x-2)(x^2-x+2)$

6) $(x+1)(2x-5)(x+2)$

B.

1) $1+\dfrac{4x+1}{x^2-1}$

2) $x-2+\dfrac{-x+3}{x^2+1}$

3) $x^2-2x+4-\dfrac{6}{x+2}$

4) $x^2-2x+7-\dfrac{9}{x+2}$

5) $x-4+\dfrac{8x+7}{x^3+2x^2+x}$

6) $x+\dfrac{-7x^2+2}{x^3+4x}$

C.

1) $(x-2)(x+2)(x^2+4)$

2) $(x-2)(x^2+2x+4)$

3) $(x-5)(x+3)$

4) $x(x-6)(x+4)$

5) $(3x+2)(2x-1)$

6) $(4x+1)(2x-3)$

7) $(x^2-1)(x^2-9)=(x-1)(x+1)(x-3)(x+3)$

8) $(2x+3)(4x^2-6x+9)$

9) $(x-1)(x+1)(x^2+2)$

10) $(x-2)(x^2+4)$

11) $(x+1)^2(x-2)$

12) $(x+1)(x-2)(x-3)$

13) $(x-3)(x-4)(x+2)$

14) $(x-1)(x-2)(x-6)$

15) $(x+2)^3(x-3)$

ALGEBRA MODULE TWO

EXPONENTS AND RADICALS

REVIEW OF PROPERTIES OF EXPONENTS

Let x and y be real numbers, and let m and n be integers.

Property		Example
1)	$x^m x^n = x^{m+n}$	$2^3 2^2 = 2^{3+2} = 2^5 = 32$
2)	$\dfrac{x^m}{x^n} = x^{m-n}$	$\dfrac{x^5}{x^2} = x^{5-2} = x^3$
3)	$x^0 = 1$	$3^0 = 1$
4)	$\dfrac{1}{y^n} = y^{-n}$	$\dfrac{1}{y^3} = y^{-3}$
5)	$(x\,y)^m = x^m y^m$	$(4x)^3 = 4^3 x^3 = 64x^3$
6)	$\left(x^m\right)^n = x^{mn}$	$\left(x^2\right)^3 = x^6$
7)	$\left(\dfrac{x}{y}\right)^m = \dfrac{x^m}{y^m}$	$\left(\dfrac{5}{y}\right)^2 = \dfrac{25}{y^2}$

THE NTH ROOT OF A REAL NUMBER

If **a** and **b** are nonnegative real numbers and **n** is a positive integer, or if **a** and **b** are negative real numbers and **n** is an odd positive integer, then the principal nth root of a is defined as $\sqrt[n]{a} = b$ if and only if $a = b^n$.

Note that even roots of positive numbers exist in pairs. (ie. $2^2 = 4$ and $(-2)^2 = 4$ so that 2 and -2 are both square roots of 4). On the other hand, $\sqrt{4} = 2$, not ± 2 because $\sqrt{4}$ denotes the principal nth root (positive) square root of 4.

Note that these roots can also be written in exponent form, so that $\sqrt{a} = a^{1/2}$, $\sqrt[3]{a} = a^{1/3}$, etc.

Examples:

1) $\sqrt{81} = 9$ because $9^2 = 81$

2) $27^{1/3} = 3$ because $3^3 = 27$

3) $\sqrt[5]{32} = 2$ because $2^5 = 32$

4) $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

5) $16^{1/4} = 2$ because $2^4 = 16$

FRACTIONAL EXPONENTS

When we write a fractional exponent such as $x^{m/n}$, we can use property 6) above to evaluate the term.

1) $8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4$

2) $8^{-2/3} = \dfrac{1}{8^{2/3}} = \dfrac{1}{4}$

3) $(-8)^{2/3} = \left(-8^{1/3}\right)^2 = (-2)^2 = 4$

4) $(-8)^{-2/3} = \dfrac{1}{(-8)^{2/3}} = \dfrac{1}{4}$

RADICALS

A. Rewrite the expression to eliminate the radicals in the denominator.

1) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ (this process is called rationalizing the denominator.)

For an expression with a sum or difference involving a radical, such as $x + b\sqrt{y}$ we can eliminate the radical in the denominator by multiplying the numerator and the denominator by the conjugate $x - b\sqrt{y}$. This is also called rationalizing. This process makes use of the difference of squares because

$$\left(x + b\sqrt{y}\right)\left(x - b\sqrt{y}\right) = x^2 - b^2 y$$

2) $$\frac{4}{3-\sqrt{2}} = \frac{4}{3-\sqrt{2}} \cdot \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{4\left(3+\sqrt{2}\right)}{9-2} = \frac{4\left(3+\sqrt{2}\right)}{7}$$

3) $$\frac{4}{\sqrt{x+h}+\sqrt{x}} = \frac{4}{\sqrt{x+h}+\sqrt{x}} \cdot \frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}-\sqrt{x}} = \frac{4\left(\sqrt{x+h}-\sqrt{x}\right)}{x+h-x} = \frac{4\left(\sqrt{x+h}-\sqrt{x}\right)}{h}$$

B. Rewrite the expression to eliminate the radicals in the numerator:

1) $$\frac{\sqrt{3}-\sqrt{5}}{2} = \frac{\sqrt{3}-\sqrt{5}}{2} \cdot \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}} = \frac{3-5}{2\left(\sqrt{3}+\sqrt{5}\right)} = \frac{-2}{2\left(\sqrt{3}+\sqrt{5}\right)} = \frac{-1}{\sqrt{3}+\sqrt{5}}$$

2) $$\frac{\sqrt{x+2}-2}{x-2} = \frac{\sqrt{x+2}-2}{x-2} \cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \frac{x+2-4}{(x-2)\left(\sqrt{x+2}+2\right)}$$
$$= \frac{x-2}{(x-2)\left(\sqrt{x+2}+2\right)} = \frac{1}{\sqrt{x+2}+2}$$

3) $$\frac{\sqrt{x^2+5}-3}{x-2} = \frac{\sqrt{x^2+5}-3}{x-2} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \frac{x^2+5-9}{(x-2)\left(\sqrt{x^2+5}+3\right)} = \frac{x^2-4}{(x-2)\left(\sqrt{x^2+5}+3\right)}$$
$$= \frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+5}+3\right)} = \frac{x+2}{\sqrt{x^2+5}+3}$$

SOLVED PROBLEMS INVOLVING EXPONENTS

Simplify the following expressions

1) $\left(7^2a^2b^3\right)\left(7^4a^9b^2\right)$

Solution: Regrouping and combining like terms.

$\left(7^2a^2b^3\right)\left(7^4a^9b^2\right) = \left(7^27^4\right)\left(a^2a^9\right)\left(b^3b^2\right) = 7^6a^{11}b^5$ (adding exponents)

2) $\dfrac{128x^3}{32x^5}$

Solution: We express 128 and 32 as powers of the common base 2.

$$\frac{128x^3}{32x^5} = \frac{2^7x^3}{2^5x^5} = 2^2x^{-2} \quad \text{(subtracting exponents)}$$

$$= \frac{4}{x^2} \qquad \left(x^{-2} = \frac{1}{x^2}\right)$$

3) $\dfrac{(2x)^{-3}}{(3y)^{-2}}$

Solution: $(2x)^{-3} = \dfrac{1}{(2x)^3}$ and $(3y)^{-2} = \dfrac{1}{(3y)^2}$

So $\dfrac{(2x)^{-3}}{(3y)^{-2}} = \dfrac{(3y)^2}{(2x)^3} = \dfrac{3^2y^2}{2^3x^3} = \dfrac{9y^2}{8x^3}$

4) $\left(3x^{-2}y^3\right)^4$

Solution: $\left(3x^{-2}y^3\right)^4 = \left(\dfrac{3y^3}{x^2}\right)^4 = \dfrac{3^4y^{12}}{x^8} = \dfrac{81y^{12}}{x^8}$

5) $\left(\dfrac{4^{-2}x^{-4}}{4^{-5}y^{-3}}\right)^5$

Solution: We simplify the inside of the parenthesis first using

$$\left.\begin{array}{ll} 4^{-2} = \dfrac{1}{4^2} & x^{-4} = \dfrac{1}{x^4} \\[2ex] \dfrac{1}{4^{-5}} = 4^5 & \dfrac{1}{y^{-3}} = y^3 \end{array}\right\}$$

So $\left(\dfrac{4^{-2}x^{-4}}{4^{-5}y^{-3}}\right)^5 = \left(\dfrac{4^5}{4^2}\,\dfrac{y^3}{x^4}\right)^5 = \left(\dfrac{4^3y^3}{x^4}\right)^5 = \dfrac{4^{15}y^{15}}{x^{20}}$

6) $3^{2n+1} \cdot 3^{n-2}$

Solution: We add the exponents and combine like terms.

$$3^{2n+1} \cdot 3^{n-2} = 3^{(2n+1)+(n-2)} = 3^{3n-1}$$

7) $\dfrac{2^{4n+1}}{2^{5n-1}}$

Solution: We subtract the exponents and combine like terms.

$$\frac{2^{4n+1}}{2^{5n-1}} = 2^{(4n+1)-(5n-1)} = 2^{(4n-5n)+(1--1)}$$

$$= 2^{-n+2}$$

$$= 2^{2-n} \text{ or } \frac{1}{2^{n-2}}$$

8) $\dfrac{4^{n+3}}{2^{n-1}}$

Solution: Change 4 to 2^2.

Thus

$$\frac{4^{n+3}}{2^{n-1}} = \frac{\left(2^2\right)^{n+3}}{2^{n-1}} = \frac{2^{2n+6}}{2^{n-1}} = 2^{(2n+6)-(n-1)}$$

$$= 2^{(2n-n)+(6--1)}$$

$$= 2^{n+7}$$

9) $\sqrt{50x^4y^8}$

Solution: Use the fact that $\sqrt{abc} = \sqrt{a}\,\sqrt{b}\,\sqrt{c}$.

$$\sqrt{50x^4y^8} = \sqrt{50}\,\sqrt{x^4}\,\sqrt{y^8}$$

$$= \sqrt{25}\,\sqrt{2}\,\sqrt{x^4}\,\sqrt{y^8} \quad \text{(NOTE: } \sqrt{x^4} = \sqrt{\left(x^2\right)^2} = x^2$$

$$\sqrt{y^8} = \sqrt{\left(y^4\right)^2} = y^4\text{)}$$

$$= 5\sqrt{2}\,x^2y^4$$

$$= \sqrt{2}\left(5x^2y^4\right)$$

10) $\sqrt[3]{128x^4y^7}$

Solution: Use the fact that $\sqrt[3]{abc} = \sqrt[3]{a}\,\sqrt[3]{b}\,\sqrt[3]{c}$.

Also $\sqrt[3]{x^3} = x \qquad \sqrt[3]{y^6} = \sqrt[3]{\left(y^2\right)^3} = y^2$ and $128 = 2^7 = 2^6 \cdot 2^1$

So

$$\sqrt[3]{128x^4y^7} = \sqrt[3]{128}\,\sqrt[3]{x^4}\,\sqrt[3]{y^7}$$
$$= \left(\sqrt[3]{2^6}\,\sqrt[3]{2}\right)\left(\sqrt[3]{x^3}\,\sqrt[3]{x}\right)\left(\sqrt[3]{y^6}\,\sqrt[3]{y}\right)$$
$$= \left(\sqrt[3]{\left(2^2\right)^3}\,\sqrt[3]{2}\right)\left(x\sqrt[3]{x}\right)\left(y^2\sqrt[3]{y}\right)$$
$$= \left(2^2xy^2\right)\sqrt[3]{x}\,\sqrt[3]{2}\,\sqrt[3]{y} = \left(4xy^2\right)\sqrt[3]{2xy}$$

Solve for k.

1) $3^{n+1}\cdot 3^k = 3^{n-2}$

Solution: $3^k = \dfrac{3^{n-2}}{3^{n+1}}$ (Divide by 3^{n+1}).

$3^k = 3^{-2-1} = 3^{-3}$. (The n's cancel out.) Comparing exponents we see $k = -3$.

2) $\dfrac{2^{n+3}}{2^{k+1}} = 2^{2n-1}$

Solution: Subtracting exponents on the left side gives

$2^{(n+3)-(k+1)} = 2^{2n-1}$

$= 2^{n-k+2} = 2^{2n-1}$

Setting exponents equal and solving for k.

$n - k + 2 = 2n - 1$

$(n+2) - (2n-1) = k$

$-n + 3 = k$

or $k = -n + 3 = 3 - n$

PROBLEMS

A. Evaluate:

1) $9^{3/2}$ 2) $(-27)^{2/3}$ 3) $8^{-4/3}$ 4) $16^{3/4}$ 5) $(-32)^{-2/5}$

B. Solve for x:

1) $x^{2/3} = 4$ 2) $x^{3/2} = 8$ 3) $x^{3/4} = 8$ 4) $x^{3/5} = -8$

C. Eliminate all radicals from the numerator and simplify:

1) $\dfrac{\sqrt{2}}{2}$ 2) $\dfrac{4-\sqrt{2}}{5}$ 3) $\dfrac{\sqrt{4+h}-2}{h}$ 4) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

5) $\dfrac{\sqrt{x+7}-3}{x-2}$ 6) $\dfrac{\sqrt{x^2+16}-5}{x-3}$ 7) $\dfrac{\sqrt{x^3-2}-5}{x-3}$ 8) $\dfrac{\sqrt{x^3+8}-4}{x^3+4x-16}$

D. Simplify:

1) $(9x^5)(27x^2)$ 2) $\dfrac{1024x^{-4}y^{-2}}{64x^{-6}y^3}$ 3) $(3x^2y^{-3})^{-2}$

4) $\left(\dfrac{25x^{-3}}{625y^{-2}}\right)^{-2}$ 5) $\dfrac{(2x^{-1}y^3)^3(4x^2y^{-3})^2}{(32x^{-5}y^4)^3}$ 6) $\dfrac{(4x^{-3})^{-2}}{(5y^{-2})^{-4}}$

7) $\dfrac{5^{2n}\cdot 5^{n+1}}{5^{4n-2}}$ 8) $\dfrac{3^{n+1}\,9^{n-1}}{(27)^{n+2}}$ 9) $\dfrac{(32)^{n+1}}{(64)^{n-1}}$ 10) $\sqrt[3]{54x^5y^{13}}$

E. Solve for k. 1) $3^{n+2}\,3^{k-1} = 3^{n-3}$ 2) $\dfrac{2^{n+1}}{2^{k-1}} = 2^{3n}$

ANSWERS

A. 1) 27 2) 9 3) $\dfrac{1}{16}$ 4) 8 5) $\dfrac{1}{4}$

B. 1) $x = 4^{3/2} = \pm 8$ 2) $x = 8^{2/3} = 4$ 3) $x = 8^{4/3} = 16$ 4) $x = (-8)^{5/3} = -32$

C. 1) $\dfrac{1}{\sqrt{2}}$ 2) $\dfrac{14}{5(4+\sqrt{2})}$ 3) $\dfrac{1}{\sqrt{4+h}+2}$ 4) $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$ 5) $\dfrac{1}{\sqrt{x+7}+3}$

6) $\dfrac{x+3}{\sqrt{x^2+16}+5}$ 7) $\dfrac{x^2+3x+9}{\sqrt{x^3-2}+5}$ 8) $\dfrac{x^2+2x+4}{(x^2+2x+8)(\sqrt{x^3+8}+4)}$

D. 1) 3^5x^7 or $243x^7$ 2) $\dfrac{2^4x^2}{y^5}$ or $\dfrac{16x^2}{y^5}$ 3) $\dfrac{y^6}{9x^4}$ 4) $\dfrac{625x^6}{y^4}$

5) $\dfrac{x^{16}}{256y^9}$ 6) $\dfrac{625x^6}{16y^8}$ 7) 5^{3-n} or $\dfrac{1}{5^{n-3}}$

8) $\dfrac{1}{3^7}$ or $\dfrac{1}{2187}$ 9) 2^{-n+11} or $\dfrac{1}{2^{n-11}}$ 10) $(3xy^4)\sqrt[3]{2x^2y}$

E. 1) $k = -4$ 2) $k = -2n+2$

ALGEBRA MODULE THREE

RATIONAL FUNCTIONS

The properties for manipulations of fractions apply also to fractions of polynomials which are known as rational functions.

1) ADDITION: $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

2) MULTIPLICATION: $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$

3) DIVISION: $\frac{a/b}{c/d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$ and also $\frac{a/b}{c}=\frac{a}{b}\cdot\frac{1}{c}=\frac{a}{bc}$

Perform the indicated operations and simplify the result as much as possible:

1) $$\frac{x+1}{x}+\frac{x}{x-1}=\frac{(x+1)(x-1)+x(x)}{x(x-1)}=\frac{x^2-1+x^2}{x^2-x}=\frac{2x^2-1}{x^2-x}$$

2) $$\frac{x-1}{x^2-2x}+\frac{2}{x^2-x-2}=\frac{x}{x(x-2)}+\frac{2}{(x-2)(x+1)}$$

Note that both fractions contain the factor $x-2$. Therefore, in finding the common denominator for the expression, we use the lowest common denominator (the LCD).

$$\frac{x-1}{x(x-2)}\cdot\frac{x+1}{x+1}+\frac{2}{(x-2)(x+1)}\cdot\frac{x}{x}=\frac{x^2-1+2x}{x(x-2)(x+1)}=\frac{x^2+2x-1}{x(x-2)(x+1)}$$

3) $$\frac{x^2}{x+2}\cdot\frac{x+2}{x^2-3x}=\frac{x^2(x+2)}{(x+2)x(x-3)}=\frac{x}{x-3}$$

4) $$\frac{\frac{x^2-4}{x}}{\frac{x-2}{x^2}}=\frac{(x-2)(x+2)}{x}\cdot\frac{x^2}{x-2}=(x+2)x=x^2+2x$$

5) $$\frac{\frac{1}{5}-\frac{1}{x}}{x-5}=\frac{\frac{x-5}{5x}}{x-5}=\frac{x-5}{5x(x-5)}=\frac{1}{5x}$$

6) $$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\frac{\frac{x-(x+h)}{(x+h)(x)}}{h}=\frac{x-x-h}{(x+h)(x)}\cdot\frac{1}{h}=\frac{-h}{(x+h)(x)(h)}=\frac{-1}{(x+h)(x)}$$

PROBLEMS

Perform the indicated operation and write the answer in factored form where possible:

1) $\dfrac{x}{x+3}-\dfrac{x-2}{x}$

2) $\dfrac{2}{x^2-4}+\dfrac{1}{x^2+3x+2}$

3) $\dfrac{1}{x+2}+\dfrac{4}{x^2-x-6}$

4) $\dfrac{x^2-9}{x^3}\cdot\dfrac{x}{x^2+4x+3}$

5) $\dfrac{\frac{x^2+2x+1}{x+2}}{\frac{x+1}{x^2+x-2}}$

6) $\dfrac{\frac{x^2-3x-4}{x}}{x-4}$

7) $\dfrac{\frac{1}{7}-\frac{1}{x}}{x-7}$

8) $\dfrac{\frac{1}{x}-\frac{1}{3}}{x^2-9}$

9) $\dfrac{\frac{1}{4}-\frac{1}{x+2}}{x^2-3x+2}$

10) $\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

11) $\dfrac{\frac{1}{6}+\frac{1}{x+3}}{x^2+5x-36}$

ANSWERS

1) $\dfrac{-x+6}{x(x+3)}$

2) $\dfrac{3x}{(x-2)(x+2)(x+1)}$

3) $\dfrac{x+1}{(x-3)(x+2)}$

4) $\dfrac{x-3}{x^2(x+1)}$

5) $(x+1)(x-1)$

6) $\dfrac{x+1}{x}$

7) $\dfrac{1}{7x}$

8) $\dfrac{-1}{3x(x+3)}$

9) $\dfrac{1}{4(x+2)(x-1)}$

10) $\dfrac{-2x-h}{(x+h)^2x^2}$

11) $\dfrac{1}{6(x+3)(x-4)}$