Dawson College: **Calculus III**: 201-BZF-05 S01 September 24, 2014

Last Name: SOLUTIONS

First Name:

Student ID:

Test 1

Question 1. *(5 marks)* Use the **Monotonic Sequence Theorem** to show that the following sequence converges

$$\left\{\frac{\ln n}{n}\right\}_{n=3}^{\infty}$$

LET $f(x) = \dfrac{\ln x}{x}$

$$f'(x) = \frac{\frac{1}{x}\cdot x - (\ln x)1}{x^2} = \frac{1-\ln x}{x^2} < 0 \quad \text{WHEN } x \geq 3 > e$$

$\therefore\ a_n = \dfrac{\ln n}{n}$ IS DECREASING.

$\Rightarrow \quad a_n \leq a_3 = \dfrac{\ln 3}{3}$ FOR ALL $n \geq 3$

NOTICE $a_n = \dfrac{\ln n}{n} > 0$ WHEN $n \geq 3 > e$

SO $0 < a_n \leq \dfrac{\ln 3}{3}$

$\{a_n\}$ IS A BOUNDED MONOTONIC SEQUENCE $\therefore$ IT CONVERGES

Question 2. Determine if the following series converge or diverge.

a) *(5 marks)* $\sum_{n=1}^{\infty} \frac{\sqrt{2n+3}}{3n^2+2n+5}$

LET $b_n = \frac{1}{n^{3/2}}$

$a_n = \frac{\sqrt{2n+3}}{3n^2+2n+5}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{2n+3}}{3n^2+2n+5} \cdot \frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty} \frac{n \cdot \sqrt{n}\,\sqrt{2n+3}}{3n^2+2n+5} = \lim_{n\to\infty} \frac{n\sqrt{2n^2+3n}}{3n^2+2n+5}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2+3/n}}{3+2/n+5/n^2} = \frac{\sqrt{2}}{3} > 0$$

SINCE $\sum b_n$ CONVERGES (p-SERIES, $p = 3/2 > 1$)

SO DOES $\sum a_n$ BY LIMIT COMPARISON TEST.

b) *(3 marks)* $\sum_{n=1}^{\infty} \ln\left(\frac{3n^2+2}{4n^2+6}\right)$

$$\lim_{n\to\infty} \ln\left(\frac{3n^2+2}{4n^2+6}\right) = \ln\left(\lim_{n\to\infty} \frac{3n^2+2}{4n^2+6}\right) = \ln\left(\frac{3}{4}\right) \neq 0$$

$\therefore$ THE SERIES DIVERGES BY THE TEST FOR DIVERGENCE.

c) *(5 marks)* $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^3}$

LET $f(x) = \dfrac{1}{x(\ln x)^3}$. f IS A POSITIVE, *DECREASING, CONTINUOUS FUNCTION ON $[3,\infty)$

$$\int_3^\infty f(x)\,dx = \lim_{b\to\infty} \int_3^b \frac{1}{x(\ln x)^3}\,dx = I$$

$\boxed{\begin{array}{l}\text{LET } u = \ln x \\ du = \frac{1}{x}\,dx\end{array}}$

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = \frac{u^{-2}}{-2} + C = -\frac{1}{2(\ln x)^2} + C$$

$$\therefore \quad I = \lim_{b\to\infty}\left[-\frac{1}{2(\ln x)^2}\right]_3^b = \lim_{b\to\infty}\left(-\frac{1}{2(\ln b)^2} + \frac{1}{2(\ln 3)^2}\right)$$

$$= 0 + \frac{1}{2(\ln 3)^2} = \frac{1}{2(\ln 3)^2}$$

SINCE THE IMPROPER INTEGRAL $\int_3^\infty f(x)\,dx$ CONVERGES SO DOES $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^3}$ BY THE INTEGRAL TEST.

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* DECREASING: SINCE BOTH x AND $\ln x$ ARE INCREASING ON $[3,\infty)$

$f(x) = \dfrac{1}{x(\ln x)^2}$ IS DECREASING

OR

$$f'(x) = -\left[x(\ln x)^3\right]^{-2} \cdot \left[1(\ln x)^3 + x \cdot 3(\ln x)^2 \cdot \frac{1}{x}\right]$$

$$= -\frac{(\ln x)^3 + 3(\ln x)^2}{\left(x(\ln x)^3\right)^2} < 0 \quad \text{ON } [3,\infty)$$

Question 3. Find the sum of each of the following series if they converge.

a) *(5 marks)* $\sum_{n=1}^{\infty}\left[\cos\left(\frac{n+1}{n^2}\right)-\cos\left(\frac{n+2}{(n+1)^2}\right)\right]$

NEXT TERM

$$S_n = \left(\cos\frac{2}{1} - \cancel{\cos\frac{3}{2^2}}\right) + \left(\cancel{\cos\frac{3}{2^2}} - \cancel{\cos\frac{4}{3^2}}\right) + \left(\cancel{\cos\frac{4}{3^2}} - \cancel{\cos\frac{5}{4^2}}\right)$$

PREVIOUS TERM

$$+ \ldots + \left(\cancel{\cos\frac{n-1}{(n-2)^2}} - \cancel{\cos\frac{n}{(n-1)^2}}\right) + \left(\cancel{\cos\frac{n}{(n-1)^2}} - \cancel{\cos\frac{n+1}{n^2}}\right)$$

$$+ \left(\cancel{\cos\frac{n+1}{n^2}} - \cos\left(\frac{n+2}{(n+1)^2}\right)\right)$$

$$= \cos 2 - \cos\left(\frac{n+2}{(n+1)^2}\right)$$

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{n+1}{n^2}\right) - \cos\frac{n+2}{(n+1)^2}\right] = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left[\cos 2 - \cos\left(\frac{n+2}{(n+1)^2}\right)\right]$$

$$= \cos 2 - \cos\left[\lim_{n\to\infty}\frac{n+2}{(n+1)^2}\right]$$

$$= \cos 2 - \cos 0$$

$$= \cos 2 - 1 \text{ (CONVERGES)}$$

b) *(5 marks)* $\sum_{n=2}^{\infty} \frac{2^{n+1} - 4 \cdot 5^n}{7^{n-1}}$ $= \sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{7^{n-1}} - 4\frac{5^n}{7^{n-1}}\right)$

$$= \sum_{n=2}^{\infty} \frac{2^{n+1}}{7^{n-1}} - \sum_{n=2}^{\infty} 4\frac{5^n}{7^{n-1}}$$

$$= \sum_{n=2}^{\infty} 2^2\left(\frac{2}{7}\right)^{n-1} - \sum_{n=2}^{\infty} 4\cdot 5\left(\frac{5}{7}\right)^{n-1}$$

$$= \sum_{n=2}^{\infty} 4\left(\frac{2}{7}\right)^{n-1} + \underset{a_1}{4} - 4 - \left(\sum_{n=2}^{\infty} 20\left(\frac{5}{7}\right)^{n-1} + \underset{b_1}{20} - 20\right)$$

$$= \underbrace{\sum_{n=1}^{\infty} 4\left(\frac{2}{7}\right)}_{\text{GEOMETRIC}} - 4 - \left(\underbrace{\sum_{n=1}^{\infty} 20\left(\frac{5}{7}\right)^{n-1}}_{\text{GEOMETRIC}} - 20\right)$$

$|r| = \frac{2}{7} < 1$ $\therefore$ CONVERGENT

$|r| = \frac{5}{7} < 1$ $\therefore$ CONVERGENT

$$= \frac{4}{1-\frac{2}{7}} - 4 - \left(\frac{20}{1-\frac{5}{7}} - 20\right)$$

$$= \frac{4}{5/7} - 4 - \left(\frac{20}{2/7} - 20\right)$$

$$= \frac{28}{5} - 4 - \left(\frac{140}{2} - 20\right)$$

$$= \frac{8}{5} - 50$$

$$= -\frac{242}{5}$$

Question 4. *(7 marks)* Find the interval and Radius of convergence of the following power series.

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-4)^n}{3n+2}$$

LET $a_n = \frac{(-1)^n (x-4)^n}{3n+2}$

RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3n+5}\cdot\frac{3n+2}{(-1)^n(x-4)^n}\right|$$

$$= \lim_{n\to\infty}|x-4|\cdot\frac{3n+2}{3n+5} = |x-4|\cdot\frac{3}{3} = |x-4| < 1$$ (FOR ABSOLUTE CONVERGENCE)

$\therefore$ THE RADIUS OF CONVERGENCE IS $\underline{\underline{R=1}}$

NOW, $|x-4|<1 \iff -1 < x-4 < 1 \iff 3 < x < 5$

AT $\underline{x=3}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(3-4)^n}{3n+2} = \sum_{n=0}^{\infty}\frac{1}{3n+2} = \sum_{n=0}^{\infty} a_n$$ LET $b_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{3n+2}\cdot\frac{n}{1} = \frac{1}{3} > 0$$

SINCE $\sum\frac{1}{n}$ DIVERGES (p-SERIES, $p=1$) SO DOES $\sum a_n$ BY LIMIT COMPARISON TEST.

AT $x=5$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(5-4)^n}{3n+2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+2}$$ LET $b_n = \frac{1}{3n+2} > 0$

THIS IS AN ALTERNATING SERIES.

1) $b_{n+1} = \frac{1}{3n+5} < \frac{1}{3n+2} = b_n$ 2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{3n+2} = 0$

$\therefore \sum_{n=0}^{\infty}(-1)^n b_n$ CONVERGES BY ALTERNATING SERIES TEST.

$\therefore$ THE INTERVAL OF CONVERGENCE IS $(3,5]$

Question 5. *(5 marks)* Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty}(-1)^n\frac{2^n\cdot 3^{n+1}}{n^n} = \sum_{n=1}^{\infty} a_n$$

ROOT TEST:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n 2^n\cdot 3^{n+1}}{n^n}\right|} = \lim_{n\to\infty}\sqrt[n]{\frac{2^n\cdot 3^n\cdot 3}{n^n}}$$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{2^n}\cdot\sqrt[n]{3^n}\cdot\sqrt[n]{3}}{\sqrt[n]{n^n}} = \lim_{n\to\infty}\frac{2\cdot 3\cdot 3^{1/n}}{n}$$

$$= \lim_{n\to\infty}\frac{6}{n}\cdot 3^{1/n} = 0\cdot 3^0 = 0 < 1$$

$\therefore$ BY THE ROOT TEST THE SERIES $\sum a_n$ IS ABSOLUTELY CONVERGENT.

Question 6.

a) *(5 marks)* Find a power series representation for $f(x) = x^2 \arctan x$. State the radius of convergence of this series.

Hint: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ and $\arctan x = \int \frac{1}{1+x^2}\,dx$.

FOR $|x| < 1$

Use the fact $\arctan(0) = 0$ to find the constant of integration.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{FOR } |x|<1 \Rightarrow \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

FOR $|-x^2| < 1 \Leftrightarrow |x| < 1$

NOW
$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$
$$= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C \text{ FOR } |x|<1$$

$*$

$$\therefore\ x^2 \arctan x = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1} \quad \text{FOR } |x|<1$$

$$\boxed{\therefore\ x^2 \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1} \quad \text{WITH RADIUS OF CONVERGENCE } R=1}$$

$*$ $$\arctan(0) = 0 = C + \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+1}}{2n+1} = C + 0$$

$$\Rightarrow C = 0$$

b) *(5 marks)* Use part **a)** to approximate $f(0.2) = 0.04\arctan(0.2)$ to with $|\text{Error}| < 10^{-7}$. Can we use part **a)** to approximate $f(2.2)$? Why or why not?

$$x^2 \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1} \quad \text{FOR } |x|<1$$

$$\therefore f(0.2) = (0.2)^2 \arctan(0.2) = \sum_{n=0}^{\infty} \frac{(-1)^n (0.2)^{2n+3}}{2n+1}$$

$$= \frac{(0.2)^3}{1} - \frac{(0.2)^5}{3} + \frac{(0.2)^7}{5} - \frac{(0.2)^9}{7} - \cdots$$

$$b_3 = \frac{(0.2)^9}{7} \approx 7.3 \times 10^{-8} < 10^{-7}$$

NOW $b_n = \dfrac{(0.2)^{2n+3}}{2n+1}$ SO

1) $b_{n+1} = \dfrac{(0.2)^{2n+5}}{2n+3} < \dfrac{(0.2)^{2n+3}}{2n+1} = b_n$

2) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{(0.2)^{2n+3}}{2n+1} = 0$

SO WE CAN USE THE ALTERNATING SERIES ESTIMATION THEOREM.

SO $|S - S_2| < b_3 < 10^{-7}$

AND $S_4 = (0.2)^3 - \dfrac{(0.2)^5}{3} + \dfrac{(0.2)^7}{5} \approx 7.895893 \times 10^{-3}$

SO TO WITHIN $|\text{ERROR}| < 10^{-7}$

$f(0.2) \approx 7.895893 \times 10^{-3}$