Dawson College: Calculus III: 201-BZF-05 S01

October 29, 2014

Last Name: SOLUTIONS

First Name:

Student ID:

Test 2

The length of the test is 1hr and 45min. The test consistst of 9 questions. Complete any 8 of the 9 questions during the test. **Hand in the 9th question of your choice at the beginning of tomorrow's class.** You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation.** Do not use decimals unless otherwise stated.

Question 1. *(5 marks)* Use the definition of a Maclaurin series to find the Maclaurin series for the function $f(x) = \ln(1+3x)$.

$f(x) = \ln(1+3x)$

$f'(x) = 3(1+3x)^{-1}$

$f''(x) = -3^2(1+3x)^{-2}$

$f'''(x) = 3^3 \cdot 2(1+3x)^{-3}$

$f^{(4)}(x) = -3^4 \cdot 3 \cdot 2(1+3x)^{-4}$

$\Rightarrow$

$f(0) = 0$

$f'(0) = 3$

$f''(0) = -3^2$

$f'''(0) = 3^3 \cdot 2$

$f^{(4)}(0) = -3^4 \cdot 3 \cdot 2$

$\vdots$

$f^{(n)}(0) = (-1)^{n+1} \cdot 3^n \cdot (n-1)!$

For $n \geq 1$

$$\therefore T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3^n (n-1)!}{n!} x^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{n} x^n$$

Question 2. *(5 marks)* Find the limit of the vector function $\vec{\mathbf{r}}(t)$ below.

$$\lim_{t\to\infty}\left\langle 2\arctan t,\ (t+2)e^{-t},\ \frac{3t^2-1}{2t^2+4}\right\rangle = L$$

What does this limit mean geometrically?

I.F. $\frac{\infty}{\infty}$

• $\lim_{t\to\infty} 2\tan^{-1} t = 2\left(\frac{\pi}{2}\right) = \pi$ • $\lim_{t\to\infty} \frac{t+2}{e^t} \overset{(H)}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0$

• $\lim_{t\to\infty} \frac{3t^2-1}{2t^2+4} = \lim_{t\to\infty} \frac{3-1/t^2}{2+4/t^2} = \frac{3}{2}$

$\therefore\ L = (\pi, 0, 3/2)$

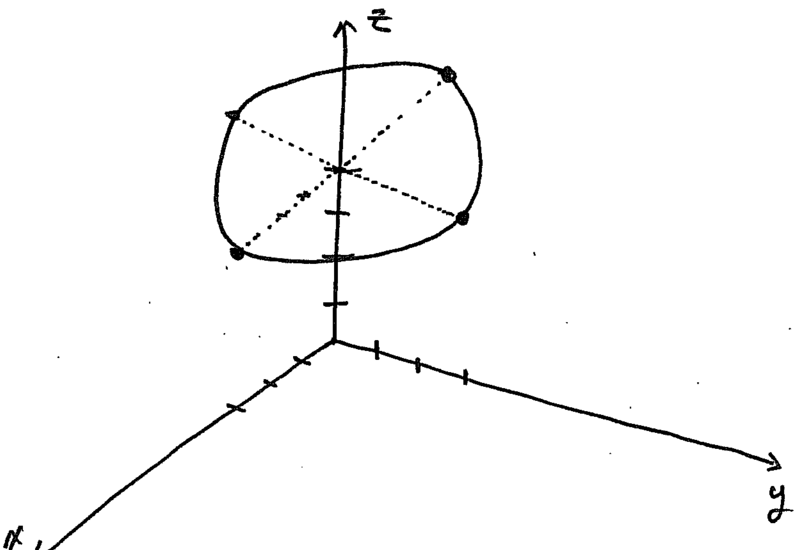
THIS MEANS THAT THE CURVE $\vec{r}(t)$ APPROACHES THE POINT $(\pi, 0, 3/2)$ AS t GETS LARGER $(t\to\infty)$

Question 3. *(5 marks)* Find the vector function that represents the curve of intersection of the sphere $x^2+y^2+z^2=25$ and the plane $z=4$. Graph this curve.

$z=4 \Rightarrow x^2+y^2+4^2=25 \Rightarrow x^2+y^2=9$

SO $x^2+y^2=9$ AND $z=4$

$\therefore\ \vec{r}(t) = \langle 3\cos t, 3\sin t, 4\rangle\ ,\quad 0\le t\le 2\pi$



Question 4. Given $\vec{\mathbf{r}}(t) = \langle \cos 8t, t, \sin 8t \rangle$

(a) *(5 marks)* Find $\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}}$ to $\vec{\mathbf{r}}(t)$ at $(1, \pi, 0)$.

$$\vec{r}\,'(t) = \langle -8\sin 8t, 1, 8\cos 8t \rangle \quad \therefore |\vec{r}\,'(t)| = \sqrt{64\sin^2 8t + 1 + 64\cos^2 8t} = \sqrt{65}$$

$$\therefore \vec{T}(t) = \frac{1}{\sqrt{65}} \langle -8\sin 8t, 1, 8\cos 8t \rangle \Rightarrow \boxed{\vec{T}(0) = \frac{1}{\sqrt{65}}(0, 1, 8)}$$

$$\vec{T}\,'(t) = \frac{1}{\sqrt{65}} \langle -64\cos 8t, 0, -64\sin 8t \rangle \quad \therefore |\vec{T}\,'(t)| = \sqrt{\frac{(-64)^2\cos^2 8t}{(\sqrt{65})^2} + \frac{(-64)^2 \sin^2 8t}{(\sqrt{65})^2}} = \boxed{\frac{64}{\sqrt{65}}}$$

$$\vec{N}(t) = \frac{\frac{-64}{\sqrt{65}} \langle \cos 8t, 0, \sin 8t \rangle}{\frac{64}{\sqrt{65}}} = -\langle \cos 8t, 0, \sin 8t \rangle$$

$$\therefore \boxed{\vec{N}(0) = \langle -1, 0, 0 \rangle}$$

$$\therefore \vec{B}(0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1/\sqrt{65} & 8/\sqrt{65} \\ -1 & 0 & 0 \end{vmatrix} = 0\vec{i} - (+8/\sqrt{65})\vec{j} + (1/\sqrt{65})\vec{k}$$

$$= \langle 0, -8/\sqrt{65}, 1/\sqrt{65} \rangle$$

(a) *(3 marks)* Find the normal plane and the osculating plane to $\vec{\mathbf{r}}(t)$ at $(1, \pi, 0)$.

$$\vec{n} = \sqrt{65}\,\vec{T}(0) = (0, 1, 8)$$

$$0x + y + 8z + d = 0$$

$$\pi + 8(0) = d$$

$$\therefore d = -\pi$$

NORMAL PLANE:

$$\boxed{y + 8z - \pi = 0}$$

$$\vec{n} = \sqrt{65}\,\vec{B}(0) = \langle 0, -8, 1 \rangle$$

$$-8y + d = 0$$

$$-8(\pi) + d = 0 \Rightarrow d = 8\pi$$

OSCULATING PLANE:

$$-8y + z + 8\pi = 0$$

Question 5. *(5 marks)* Find the curvature of $f(x) = 4 + 2x - 3x^2$ and use it to find the point on the curve with maximum curvature.

$$K(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

$$f'(x) = 2 - 6x$$

$$f''(x) = -6$$

SO $\quad K(x) = \dfrac{|-6|}{\left[1+(2-6x)^2\right]^{3/2}} = \dfrac{6}{\left[1-(2-6x)^2\right]^{3/2}}$

WHICH IS GREATEST WHEN $1-(2-6x)^2$ IS A MINIMUM

$\Leftrightarrow (2-6x) = 0 \quad \Longleftrightarrow \quad x = 1/3$

$$f(1/3) = 4 + 2(1/3) - 3(1/3)^2$$

$$= 4 + 2/3 - 1/3 = 13/3$$

$\therefore$ f HAS MAXIMUM CURVATURE AT $(1/3, 13/3)$

Question 6. *(6 marks)* Find the arc length function for

$$\vec{\mathbf{r}}(t) = \left\langle \frac{1}{2}t^2,\ 3t,\ \frac{2\sqrt{6}}{3}t^{3/2} \right\rangle$$

measured from $t = 0$ in the direction of increasing t (assume $t \geq 0$). Find the point (a,b,c) on this curve where the length from $(0,0,0)$ to (a,b,c) is 4. (Note: you do not have to reparametrize the curve with respect to arc length to solve this problem.)

$$\vec{r}\,'(t) = \langle t, 3, \sqrt{6}\, t^{1/2} \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{t^2 + 9 + 6t} = \sqrt{(t+3)^2} = t+3 \quad \text{SINCE } t \geq 0$$

$$\therefore \quad s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t (u+3)\,du = \left[\frac{u^2}{2} + 3u\right]_0^t$$

$$= \frac{t^2}{2} + 3t$$

IF $s = 4$ $\Rightarrow$

$$4 = \frac{t^2}{2} + 3t$$

$$0 = \frac{t^2}{2} + 3t - 4$$

$$0 = t^2 + 6t - 8$$

$$\therefore \quad t = \frac{-6 \pm \sqrt{6^2 - 4(1)(-8)}}{2(1)} = \frac{-6 \pm \sqrt{68}}{2} \qquad \text{BUT } t \geq 0$$

$$\text{SO} \quad t = \frac{-6 + \sqrt{68}}{2} = -3 + \sqrt{17}$$

$$\therefore \quad (a,b,c) = \left(\frac{1}{2}(-3+\sqrt{17})^2,\ 3(-3+\sqrt{17}),\ \frac{2\sqrt{6}}{3}(-3+\sqrt{17})^{3/2}\right)$$

Question 7. *(6 marks)* A moving particle has velocity function

$$\vec{v}(t) = \left\langle \sqrt{1+3t},\ \frac{1}{t+1},\ te^t \right\rangle$$

Find the position function $\vec{s}(t)$ and the acceleration function $\vec{a}(t)$ for the particle given that $\vec{s}(0) = (\frac{2}{9}, 1, 2)$.

$$\vec{s}(t) = \int v(t)dt = \left\langle \int \sqrt{1+3t}\,dt,\ \int \frac{1}{t+1}dt,\ \int te^t dt \right\rangle$$

$$\int \sqrt{1+3t}\,dt = \frac{2}{9}(1+3t)^{3/2} + C$$

$$\int \frac{1}{t+1}dt = \ln|t+1| + C$$

$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$$

$$\boxed{\begin{array}{ll} \text{LET } u = t & dv = e^t dt \\ du = dt & v = e^t \end{array}}$$

$$\vec{s}(t) = \left\langle \frac{2}{9}(1+3t)^{3/2} + C_1,\ \ln|t+1| + C_2,\ te^t - e^t + C_3 \right\rangle$$

$$\left(\frac{2}{9}, 1, 2\right) = \vec{s}(0) = \left\langle \frac{2}{9}(1+0)^{3/2} + C_1,\ \ln|0+1| + C_2,\ 0 - e^0 + C_3 \right\rangle$$

$$\therefore\ C_1 = 0,\quad C_2 = 1,\ C_3 = 3$$

$$\boxed{\vec{s}(t) = \left\langle \frac{2}{9}(1+3t)^{3/2},\ \ln|t+1| + 1,\ te^t - e^t + 3 \right\rangle}$$

$$\vec{a}(t) = \frac{d}{dt}(\vec{v}(t)) = \left\langle \frac{1}{2}(1+3t)^{-1/2} \cdot 3,\ -(t+1)^{-2},\ e^t + te^t \right\rangle$$

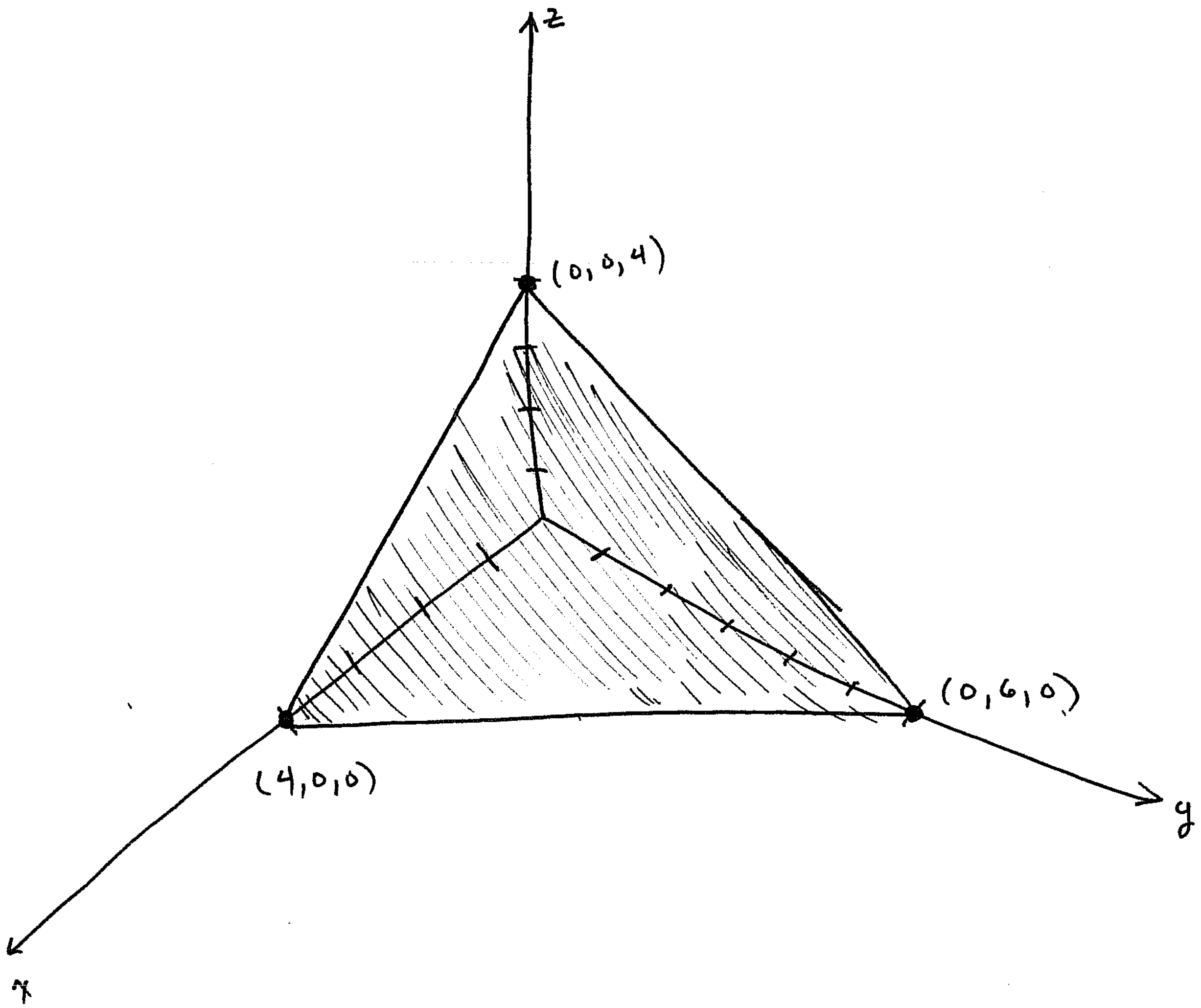
$$\boxed{\vec{a}(t) = \left\langle \frac{3}{2\sqrt{1+3t}},\ -\frac{1}{(t+1)^2},\ e^t(1+t) \right\rangle}$$

Question 8. *(4 marks)* Sketch the graph of the plane $z = 3x + 2y + 4z - 12$ in the first octant $(x, y, z \geq 0)$.

IF $x = y = 0 \Rightarrow$ [redacted] $z = 4z - 12 \Rightarrow z = 4$

$x = z = 0 \Rightarrow 0 = 2y - 12 \Rightarrow y = 6$

$y = z = 0 \Rightarrow 0 = 3x - 12 \Rightarrow x = 4$



Question 9: *(6 marks)* Determine the set of points where the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is continuous. Explain your reasoning.

$f(x,y) = \dfrac{x^2y}{x^4+y^2}$ IF $(x,y) \neq (0,0)$ AND $\dfrac{x^2y}{x^4+y^2}$ IS CONTINUOUS

AS LONG AS $(x,y) \neq (0,0)$ SO $f(x,y)$ IS CONTINUOUS ~~[illegible]~~

EVERYWHERE EXCEPT POSSIBLY AT $(0,0)$

NOW WE NEED $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ FOR CONTINUITY AT $(0,0)$.

ALONG $x=0$

$f(0,y) = \dfrac{0}{y^2} = 0 \quad (y \neq 0)$ SO $f(x,y) \to 0$ AS $(x,y) \to (0,0)$

ALONG $x=0$

ALONG $y=x^2$

$f(x,x^2) = \dfrac{x^2x^2}{x^4+(x^2)^2} = \dfrac{x^4}{2x^4} = \dfrac{1}{2} \quad (x \neq 0)$ SO $f(x,y) \to \frac{1}{2}$.

AS $(x,y) \to (0,0)$ ALONG $y=x^2$

SO $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ D.N.E. SO $f(x,y)$ IS NOT CONTINUOUS AT $(0,0)$

$\therefore f(x,y)$ IS CONTINUOUS EVERYWHERE EXCEPT $(0,0)$.

$$\kappa(t) = \frac{|\vec{\mathbf{T}}'(t)|}{|\vec{\mathbf{r}}'(t)|}$$

$$\kappa(t) = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3}$$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

$$a_T = \frac{\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t)}{|\vec{\mathbf{r}}'(t)|}$$

$$a_N = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|}$$