Dawson College: Calculus III: 201-BZF-05 S01

November 26, 2012

Last Name: SOLUTIONS.

First Name: ____________

Student ID: ____________

Test 3

The length of the test is 1hr and 45min. The test consistst of 9 questions. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Do not use decimals unless otherwise stated.

Question 1. *(5 marks)* Show that $u = \sin(x - at) + \ln(x + at)$ is a solution of the wave equation

$$u_{tt} = a^2 u_{xx}$$

$$u_t = -a\cos(x-at) + \frac{a}{x+at}$$

$$u_{tt} = -a\left[-\sin(x-at)\cdot(-a)\right] + a\left[-(x+at)^{-2}\cdot a\right]$$

$$= -a^2\sin(x-at) - \frac{a^2}{(x+at)^2}$$

$$u_x = \cos(x-at)\cdot(1) + \frac{1}{x+at}\cdot 1$$

$$u_{xx} = -\sin(x-at)\cdot 1 + \left[-(x+at)^{-2}\cdot 1\right]$$

$$= -\sin(x-at) - \frac{1}{(x+at)^2}$$

$$a^2 u_{xx} = -a^2\sin(x-at) - \frac{a^2}{(x+at)^2} = u_{tt}$$

Question 2. *(4 marks)* Find the tangent plane to the surface $z = 3(x-1)^2 + 2(y+3)^2 + 7$ at the point $(2, -2, 12)$.

LET $z = f(x,y) \Rightarrow \quad f_x = 6(x-1) \;, \; f_y = 4(y+3)$

$f_x(2,-2) = 6(2-1) = 6 \qquad f_y(2,-2) = 4(-2+3) = 4$

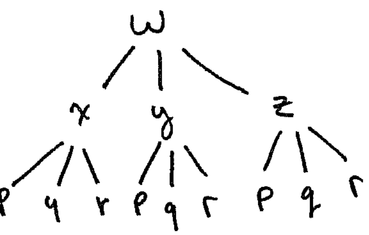
$z - z_0 = f_x(2,-2)(x - x_0) + f_y(2,-2)(y - y_0)$

$z - 12 = 6(x-2) + 4(y - (-2))$

$z = 6x - 12 + 4y + 8 + 12$

$0 = 6x + 4y - z + 8$

Question 3. *(4 marks)* Given $w = e^{x^2}\cos(z-y)$ and $x = x(p,q,r)$, $y = y(p,q,r)$, and $y = y(p,q,r)$ find an expression for $\frac{\partial w}{\partial q}$.



$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial q} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial q}$$

$$= 2xe^{x^2}\cos(x-y) \cdot \frac{\partial x}{\partial q}$$

$$+ \; e^{x^2}\sin(z-y) \cdot \frac{\partial y}{\partial q}$$

$$- \; e^{x^2}\sin(z-y)\frac{\partial z}{\partial q}$$

Question 4. Let $f(x,y) = x^2 - 3xy + 4y^3$, $\vec{v} = \vec{i} + 2\vec{j}$, and let P be the point $P(-2,0)$.

(a) *(4 marks)* Find the directional derivative of $f(x,y)$ at P in the direction of $\vec{v}$.

$f_x = 2x - 3y, \quad f_y = -3x + 12y^2 \Rightarrow \nabla f(x,y) = \langle 2x - 3y, -3x + 12y^2 \rangle$

$\nabla f(-2,0) = \langle 2(-2) - 3(0), -3(-2) + 12(0)^2 \rangle = \langle -4, 6 \rangle$

LET $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{1+4}} \langle 1, 2 \rangle = \dfrac{1}{\sqrt{5}} \langle 1, 2 \rangle$

$\therefore D_{\vec{u}} f(-2,0) = \nabla f(-2,0) \cdot \vec{u} = \langle -4, 6 \rangle \cdot \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$

$= \dfrac{-4}{\sqrt{5}} + \dfrac{12}{\sqrt{5}} = \dfrac{8}{\sqrt{5}}$

(b) *(2 marks)* Find the parametric equations of the tangent line to the function $f(x,y)$ at P in the direction of $\vec{v}$.

DIRECTION VECTOR: $\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{8}{\sqrt{5}} \rangle$ OR $\langle 1, 2, 8 \rangle$

$f(-2,0) = (-2)^2 - 0 + 0 = 4$

$\therefore$ $x = -2 + t, \quad y = 2t, \quad z = 4 + 8t, \quad t \in \mathbb{R}.$

Question 5. *(3 marks)* Using the function $f(x,y)$ and the point P from question **4** find the maximum rate of change of f at P and the direction in which is occurs.

MAXIMUM RATE OF CHANGE: $|\nabla f(-2,0)| = \sqrt{(-4)^2 + (6)^2} = \sqrt{52}$
$= 2\sqrt{13}$

DIRECTION: $\vec{v} = \nabla f(-2,0) = \langle -4, 6 \rangle$

Question 6.

For the following question your explanations must be clear for full marks.

(a) *(1 mark)* Define what is meant by a critical point (a,b) of a function $f(x,y)$.

A POINT (a,b) SUCH THAT $f_x(a,b) = 0$ AND $f_y(a,b) = 0$

OR ONE OF THESE PARTIAL DERIVATIVES DOES NOT EXIST.

(b) *(2 mark)* If (a,b) a critical point of a function $f(x,y)$ what can you say about $D_{\vec{u}}f(a,b)$ for any unit vector $\vec{u}$? Why?

$D_{\vec{u}} f(a,b) = 0$ SINCE

$$\begin{aligned} D_{\vec{u}} f(a,b) &= \nabla f(a,b) \cdot \vec{u} \\ &= \langle f_x(a,b), f_y(a,b) \rangle \cdot \vec{u} \\ &= \langle 0,0 \rangle \cdot \vec{u} \\ &= 0 \end{aligned}$$

(c) *(1 mark)* If $D_{\vec{u}}f(a,b) = 0$ for a unit vector $\vec{u}$ is (a,b) a critical point of $f(x,y)$? Why or why not?

NO, LET $f(x,y) = x^2$ THEN $f_x = 2x$ AND $f_y = 0$

LET $\vec{u} = \vec{i} = \langle 1,0 \rangle$ AND $\vec{v} = \vec{j} = \langle 0,1 \rangle$

THE $D_{\vec{u}} f(1,1) = \langle 2,0 \rangle \cdot \langle 1,0 \rangle = 2$ AND $D_{\vec{v}} f(1,1) = \langle 2,0 \rangle \cdot \langle 0,1 \rangle = 0$

SO $D_{\vec{v}} f(1,1) = f_y(1,1) = 0$ BUT $D_{\vec{u}} f(1,1) = f_x(1,1) \neq 0$ SO $(1,1)$ IS NOT A C.P.

(d) *(2 mark)* $D_{\vec{u}}f(a,b) = 0$ and $D_{\vec{v}}f(a,b) = 0$ for non-parallel unit vectors $\vec{u}$, $\vec{v}$ is (a,b) a critical point of $f(x,y)$. Why or why not? (Hint: all tangent lines at (a,b) lie in the tangent plane at (a,b)).

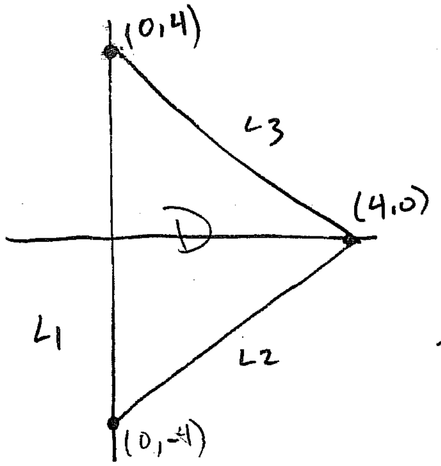
YES. SINCE $D_{\vec{u}} f(a,b) = D_{\vec{v}} f(a,b) = 0$ THE (NON-PARALLEL) TANGENT LINES AT (a,b) IN THE DIRECTIONS OF $\vec{u}$ AND $\vec{v}$ ARE HORIZONTAL $\Rightarrow$ THE TANGENT PLANE AT (a,b) IS HORIZONTAL $\Rightarrow$ $f_x(a,b) = f_y(a,b) = 0$.

Question 7. *(5 marks)* Suppose $(1,1)$, $(0,3)$ and $(-1,1)$ are critical points of the CONTINUOUS function $f(x,y)$. Given $f_{xx}(x,y) = 3xy$, $f_{yy}(x,y) = 2x+y$ and $f_{xy}(x,y) = y^2$ state whether each critical point yields a local maximum, local minimum or saddle points.

$$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$$
$$= (3xy)(2x+y) - (y^2)^2$$

(x,y)	f_{xx}	D	CONCLUSION
$(1,1)$	$3>0$	$8>0$	LOCAL MINIMUM.
$(0,3)$	0	$-81<0$	SADDLE POINT
$(-1,1)$	$-3<0$	$2>0$	LOCAL MAXIMUM

Question 8. *(6 marks)* Find the absolute maximum and minimum values of the function $f(x,y) =$ $x + y - xy$ on the closed triangular region with verticies $(0,4)$, $(4,0)$ and $(0,-4)$.



$f_x = 1 - y = 0 \Rightarrow y = 1$ $\quad f_y = 1 - x = 0 \Rightarrow x = 1$

$\therefore (1,1)$ IS THE ONLY C.P. (IT IS INSIDE D)

$f(1,1) = 1$

ALONG L_1: $x = 0 \Rightarrow f(0,y) = y \qquad -4 \le y \le 4$

MAX: $f(0,4) = 4$, MIN: $f(0,-4) = -4$

ALONG L_2: $y = x - 4 \Rightarrow f(x, x-4) = x + (x-4) - x(x-4)$

$= x + x - 4 - x^2 + 4x$

$= -x^2 + 6x - 4 \qquad 0 \le x \le 4$

NOW $\frac{d}{dx}[-x^2 + 6x - 4] = -2x + 6 = 0 \Rightarrow x = 3$

AND SO $f(3,-1) = -(3)^2 + 6(3) - 4 = 5$

$f(0,-4) = -4$

$f(4,0) = 4$

ALONG L_3: $y = -x + 4 \Rightarrow f(x, -x+4) = x + (-x+4) - x(-x+4)$

$= x - x + 4 + x^2 - 4x$

$= x^2 - 4x + 4 \quad$ ON $0 \le x \le 4$

NOW $\frac{d}{dx}[x^2 - 4x + 4] = 2x - 4 = 0 \Rightarrow x = 2$

SO $f(2,2) = (2)^2 - 4(2) + 4 = 0$

$f(0,4) = 4$

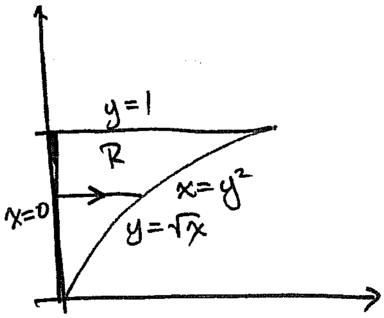
$f(4,0) = 4$

$\therefore$ ABS MAX: $f(3,-1) = 5 \qquad$ ABS MIN: $f(0,-4) = -4$

Question 9. *(6 marks)* Carefully sketch the region R in the xy-plane bounded by the lines $x = 0$, $y = 1$ and the curve $y = \sqrt{x}$. Evaluate

$$\iint_R e^{x/y^2}\, dA$$

(Hint: carefully choose your order of integration.)



$R = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y^2\}$

$$\therefore \iint_R e^{x/y^2}\, dA = \int_0^1 \int_0^{y^2} e^{x/y^2}\, dx\, dy$$

Now $\int e^{x/y^2}\, dx$

LET $u = \dfrac{x}{y^2}$

$du = \dfrac{1}{y^2}\, dy$

$y^2\, du = dy$

$= \int e^u y^2\, du$

$= y^2 e^u + C = y^2 e^{x/y^2} + C$

So $\int_0^1 \int_0^{y^2} e^{x/y^2}\, dx\, dy = \int_0^1 \left[y^2 e^{x/y^2} \right]_{x=0}^{x=y^2} dy$

$= \int_0^1 y^2 e^{y^2/y^2} - y^2 e^{0/y^2}\, dy = \int_0^1 y^2 e - y^2\, dy$

$= \int_0^1 (e-1)\, y^2\, dy = \left. \dfrac{(e-1)\, y^3}{3} \right|_0^1 = \dfrac{e-1}{3} \cdot 1^3 - 0$

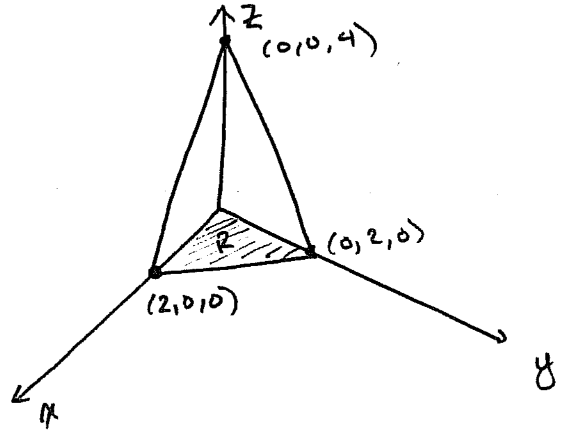
$= \dfrac{e-1}{3}$

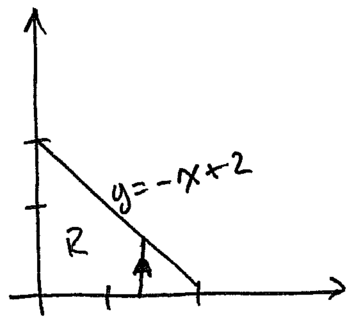
Question 10. *(5 marks)* Sketch the graph (in three dimensions) of the graph of the plane $z = 4 - 2x - 2y$ in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$). (Hint: find all the intercepst with the x, y and z-axis). Find the volume of the solid in the first octant bounded above by this plane and below by the xy-plane.

$x = y = 0 \Rightarrow z = 4$

$x = z = 0 \Rightarrow y = 2$

$y = z = 0 \Rightarrow x = 2$





$R = \{(x,y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq -x+2\}$

$\therefore\ V = \iint_R 4 - 2x - 2y \, dA$

$$= \int_0^2 \int_0^{-x+2} 4 - 2x - 2y \, dy\, dx = \int_0^2 \left[4y - 2xy - \frac{2y^2}{2} \right]_{y=0}^{y=-x+2} dx$$

$$= \int_0^2 [4(-x+2) - 2x(-x+2) - (-x+2)^2] - [0] \, dx$$

$$= \int_0^2 -4x + 8 + 2x^2 - 4x - (x^2 - 4x + 4) \, dx$$

$$= \int_0^2 x^2 - 4x + 4 \, dx = \left[\frac{x^3}{3} - 2x^2 + 4x \right]_0^2$$

$$= \left[\frac{2^3}{3} - 2(2)^2 + 4(2) \right] - [0] = \frac{8}{3} - 8 + 8 = \frac{8}{3}$$