Dawson College: **Calculus II**: 201-203-DW S03 November 15, 2012

Last Name: SOLUTIONS

First Name:

Student ID:

Quiz 3

Question 1. *(6 marks)* Verify that

$$y = \frac{C_1}{x} + C_2 \frac{\ln x}{x}$$

is a solution of the differential equation:

$$x^2 y'' + 3xy' + y = 0$$

$$y' = -\frac{C_1}{x^2} + C_2\left[\frac{\frac{1}{x}\cdot x - 1\cdot \ln x}{x^2}\right] = -\frac{C_1}{x^2} + C_2\left[\frac{1-\ln x}{x^2}\right]$$

$$y'' = \frac{2C_1}{x^3} + C_2\left[\frac{-\frac{1}{x}\cdot x^2 - 2x(1-\ln x)}{x^4}\right] = \frac{2C_1}{x^3} + C_2\left[\frac{-x-2x+2x\ln x}{x^4}\right]$$

$$= \frac{2C_1}{x^3} + C_2\left[\frac{-3x+2x\ln x}{x^4}\right]$$

$$\therefore \text{LHS} = x^2y'' + 3xy' + y = x^2\left[\frac{2C_1}{x^3} - \frac{3C_2 x}{x^4} + \frac{2C_2 x\ln x}{x^4}\right] + 3x\left[-\frac{C_1}{x^2} + \frac{C_2}{x^2} - \frac{C_2 \ln x}{x^2}\right] + \frac{C_1}{x} + C_2\frac{\ln x}{x}$$

$$= \frac{2C_1}{x} - \frac{3C_2}{x} + \frac{2C_2\ln x}{x} - \frac{3C_1}{x} + \frac{3C_2}{x} - \frac{3C_2\ln x}{x} + \frac{C_1}{x} + \frac{C_2\ln x}{x}$$

$$= \left(\frac{2C_1}{x} - \frac{3C_1}{x} + \frac{C_1}{x}\right) + \left(-\frac{3C_2}{x} + \frac{3C_2}{x}\right) + \left(\frac{2C_2\ln x}{x} - \frac{3C_2\ln x}{x} + \frac{C_2\ln x}{x}\right)$$

$$= 0 + 0 + 0 = 0$$

RHS $= 0$ $\therefore$ LHS $=$ RHS

y IS A SOLUTION OF THE D.E.

Question 2. *(6 marks)* Find the solution of the differential equation

$$y' = \frac{xy^2}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = y^2\left(\frac{x}{\sqrt{1+x^2}}\right) \Rightarrow \quad \frac{1}{y^2}\,dy = \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\therefore \underbrace{\int y^{-2}\,dy}_{I_1} = \underbrace{\int \frac{x}{\sqrt{1+x^2}}\,dx}_{I_2}$$

$$I_1 = -y^{-1} + C_1$$

$$I_2 = \int \frac{x}{\sqrt{u}}\,\frac{du}{2x} = \frac{1}{2}\int u^{-1/2}\,du$$

$$\boxed{\begin{aligned} &\text{LET } u = 1+x^2 \\ &du = 2x\,dx \\ &\frac{du}{2x} = dx \end{aligned}}$$

$$= \frac{1}{2}\,\frac{u^{1/2}}{1/2} + C_2 = \sqrt{1+x^2} + C_2$$

$$\therefore \quad -\frac{1}{y} = \sqrt{1+x^2} + C_3$$

$$-1 = y\left[\sqrt{1+x^2} + C_3\right]$$

$$\therefore \quad y = -\frac{1}{\sqrt{1+x^2} + C_3}$$

Question 3. *(6 marks)* Find the solution of the initial value problem

$$y' = 3xy - 2x; \quad y(0) = 1$$

$$\frac{dy}{dx} = x(3y-2) \Rightarrow \frac{1}{3y-2}\,dy = x\,dx$$

$$\Rightarrow \int \frac{1}{3y-2}\,dy = \int x\,dx$$

$$\frac{1}{3}\ln|3y-2| + C_1 = \frac{x^2}{2} + C_2$$

$$\therefore \ln|3y-2| = \frac{3x^2}{2} + C_3$$

$$e^{\ln|3y-2|} = e^{\frac{3x^2}{2} + C_3}$$

$$|3y-2| = e^{\frac{3x^2}{2}} \cdot e^{C_3}$$

$$|3y-2| = |C_4|\,e^{3x^2/2}$$

$$3y-2 = \pm|C_4|\,e^{3x^2/2}$$

$$3y-2 = C_5\,e^{3x^2/2}$$

$$3y = C_5\,e^{3x^2/2} + 2$$

$$y = C_6\,e^{3x^2/2} + \frac{2}{3}$$

$$y(0) = 1 \Rightarrow 1 = C_6\,e^{0^2/2} + \frac{2}{3}$$

$$\therefore 1 = C_6\,e^0 + \frac{2}{3}$$

$$\frac{1}{3} = C_6 \cdot 1$$

$$\therefore C_6 = \frac{1}{3}$$

$$\therefore \quad y = \frac{1}{3}e^{3x^2/2} + \frac{2}{3}$$

Question 4. *(6 marks)* Find the fourth taylor, $P_4(x)$, polynomial of $f(x) = e^{2x}$ at $x = 1$.

$$f(x) = e^{2x} \qquad f(1) = e^2$$
$$f'(x) = 2e^{2x} \qquad f'(1) = 2e^2$$
$$f''(x) = 4e^{2x} \quad \Rightarrow \quad f''(1) = 4e^2$$
$$f'''(x) = 8e^{2x} \qquad f'''(1) = 8e^2$$
$$f^{(4)}(x) = 16e^{2x} \qquad f^{(4)}(1) = 16e^2$$

$$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$$= e^2 + 2e^2(x-1) + \frac{4e^2}{2}(x-1)^2 + \frac{8e^2}{3\cdot 2\cdot 1}(x-1)^3 + \frac{16e^2}{4\cdot 3\cdot 2\cdot 1}(x-1)^4$$

$$= e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4}{3}e^2(x-1)^3 + \frac{2e^2}{3}(x-1)^4$$

Question 5. *(6 marks)* Find the nth taylor, of $f(x) = \frac{1}{1+x}$ at $x = 0$.

$$f(x) = (1+x)^{-1} \qquad f(0) = 1$$

$$f'(x) = -(1+x)^{-2} \qquad f'(0) = -1$$

$$f''(x) = (-1)(-2)(1+x)^{-3} \qquad f''(0) = 2$$

$$f'''(x) = (-1)(-2)(-3)(1+x)^{-4} \qquad f'''(0) = -3!$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = (-1)(-2)(-3)(-4)\ldots(-n)(1+x)^{-n-1} \qquad f^{(n)}(0) = (-1)^n n!$$

$$= (-1)^n n!\,(1+x)^{-n-1}$$

$$\therefore \quad P_n(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= 1 - 1x + \frac{2}{2!}x^2 - \frac{3!}{3!}x^3 + \ldots + \frac{(-1)^n n!}{n!}(x)^n$$

$$= 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n$$