Dawson College: Calculus II - Social Science: 201-203-DW S03

September 28, 2012

Last Name: SOLUTIONS

First Name:

Student ID:

Test 1

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**.

Question 1. Find the following antiderivatives:

(a) *(3 marks)* $\int \frac{x+\pi x^2-4}{x^2}\,dx = \int \frac{1}{x} + \pi - 4x^{-2}\,dx$

$= \ln|x| + \pi x - \frac{4x^{-1}}{-1} + C$

$= \ln|x| + \pi x + \frac{4}{x} + C$

(b) *(4 marks)* $\int \frac{e^{-1/x^2}}{x^3}\,dx$

$= \int \frac{e^u}{x^3} \cdot \frac{x^3}{2}\,du = \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^u + C = \frac{1}{2}e^{-1/x^2} + C$

LET $u = -\frac{1}{x^2} = -x^{-2}$

$du = 2x^{-3}\,dx$

$\frac{x^3}{2}\,du = dx$

(c) *(5 marks)* $\int (x^2+1)\sqrt{x+2}\,dx$

LET $u = x+2$

$du = dx$

$\Rightarrow x = u-2$

$$= \int \left[(u-2)^2 + 1 \right] \sqrt{u}\, du$$

$$= \int \left[u^2 - 4u + 4 + 1 \right] \sqrt{u}\, du = \int (u^2 - 4u + 5) u^{1/2}\, du$$

$$= \int \left(u^{5/2} - 4u^{3/2} + 5u^{1/2} \right) du = \frac{u^{7/2}}{7/2} - \frac{4u^{5/2}}{5/2} + \frac{5u^{3/2}}{3/2} + C$$

$$= \frac{2}{7}(x+2)^{7/2} - \frac{8}{5}(x+2)^{5/2} + \frac{10}{3}(x+2)^{3/2} + C$$

(d) *(4 marks)* $\int \sin(2x)\cos(2x)\,dx$

LET $u = \sin(2x)$

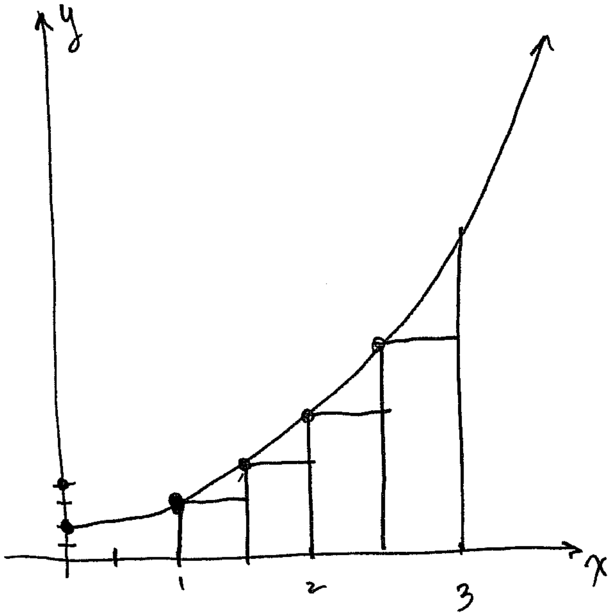
$du = 2\cos(2x)dx$

$\frac{du}{2\cos(2x)} = dx$

$$= \int u \cos 2x \frac{du}{2\cos 2x} = \frac{1}{2} \int u\, du$$

$$= \frac{1}{2} \frac{u^2}{2} + C = \frac{1}{4} \sin^2(2x) + C$$

Question 2. *(4 marks)* Sketch the graph of $f(x) = x^2 + 2$ (from 1 to 3). Approximate the area under $f(x)$ from $x = 1$ to $x = 3$ by using 4 approximating rectangles and choosing the representative points to be the left endpoints of the subintervals. Add the approximating rectangles to your sketch of $f(x)$. Would this approximation give you an upper bound or a lower bound on the exact area under $f(x)$ from $x = 1$ to $x = 3$.



$\Delta x = \frac{3-1}{4} = \frac{1}{2}$

$$L_4 = \frac{1}{2}f(1) + \frac{1}{2}f(1.5) + \frac{1}{2}f(2) + \frac{1}{2}f(2.5)$$

$$= \frac{1}{2}\left[1^2+2\right] + \frac{1}{2}\left[(1.5)^2+2\right] + \frac{1}{2}\left[2^2+2\right] + \frac{1}{2}\left[(2.5)^2+2\right]$$

$$= 10.75$$

THIS GIVES US AN ~~UPPER~~ LOWER BOUND ON THE EXACT AREA UNDER f(x) FROM $x=1$ TO $x=3$ THAT IS

AREA $<$ 10.75

Question 3. *(4 marks)* Given $f'(x) = -xe^{x^2-1}$ and $f(1) = 2$ find $f(x)$.

$$f(x) = \int -xe^{x^2-1}\,dx$$

LET $u = x^2 - 1$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$$= \int -xe^{u}\,\frac{du}{2x}$$

$$= -\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{u} + C$$

$$= -\frac{1}{2}e^{x^2-1} + C$$

NOW $2 = f(1) = -\frac{1}{2}e^{1^2-1} + C = -\frac{1}{2}e^{0} + C = -\frac{1}{2} + C$

$\therefore\ 2 + \frac{1}{2} = \frac{5}{2} = C$

SO $f(x) = -\frac{1}{2}e^{x^2-1} + \frac{5}{2}$

Question 4.

(a) *(5 marks)* Use the definition of the definite integral to find

$$\int_1^2 x^2 - 2x\,dx$$

$\bullet\ \Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$ $\quad\bullet\ x_i = a + i\Delta x = 1 + i\left(\frac{1}{n}\right) = 1 + \frac{i}{n}$

$\bullet\ f(x_i) = (x_i)^2 - 2(x_i) = \left(1+\frac{i}{n}\right)^2 - 2\left(1+\frac{i}{n}\right) = 1 + \frac{2i}{n} + \frac{i^2}{n^2} - 2 - \frac{2i}{n}$

$= \frac{i^2}{n^2} - 1$

$\bullet\ f(x_i)\Delta x = \left(\frac{i^2}{n^2} - 1\right)\left(\frac{1}{n}\right) = \frac{i^2}{n^3} - \frac{1}{n}$

$\bullet\ \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{i^2}{n^3} - \frac{1}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n}\sum_{i=1}^{n} 1$

$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{n}\cdot n = \frac{1}{6}\cdot\frac{2n^2+3n+1}{n^2} - 1$

$\therefore \int_1^2 x^2 - 2x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{1}{6}\cdot\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 1\right]$

$= \frac{1}{6}\cdot(2+0+0) - 1 = \frac{1}{3} - 1 = -\frac{2}{3}$

(b) *(1 mark)* Use the Fundamental Theorem of Calculus (Evaluation Theorem) to check your answer from part **(a)**.

$\int_1^2 x^2 - 2x\,dx = \left[\frac{x^3}{3} - x^2\right]_1^2 = \left[\frac{(2)^3}{3} - (2)^2\right] - \left[\frac{(1)^3}{3} - (1)^2\right]$

$= \frac{8}{3} - 4 - \frac{1}{3} + 1 = \frac{7}{3} - 3 = -\frac{2}{3}$

Question 5. *(5 marks)* Evaluate

$$\int_0^1 \frac{x^2+2x}{x^3+3x^2+1}\,dx$$

LET $u = x^3+3x^2+1$

$du = (3x^2+6x)\,dx$

$\frac{du}{3(x^2+2x)} = dx$

IF $x=0 \Rightarrow u = 0^3+3(0)^2+1 = 1$

$x=1 \Rightarrow u = 1^3+3(1)^2+1 = 5$

$$= \int_1^5 \frac{x^2+2x}{u} \frac{du}{3(x^2+2x)}$$

$$= \frac{1}{3}\int_1^5 \frac{1}{u}\,du$$

$$= \frac{1}{3}\left[\ln|u|\right]_1^5$$

$$= \frac{1}{3}\left[\ln|5| - \ln|1|\right]$$

$$= \frac{1}{3}\ln|5| - \frac{1}{3}(0)$$

$$= \frac{1}{3}\ln|5|$$

Question 6. *(5 marks)* The total vacancy rate (in percent) of offices in Manhattan from 2000 to 2006 is approximated by the function

$$R(t) = 0.032t^4 - 0.26t^3 - 0.478t^2 + 5.82t + 3.8 \quad (0 \le t \le 6)$$

where t is measured in years, with $t = 0$ corresponding to 2000. What was the average vacancy rate of offices in Manhattan over the period from 2000 through 2006?

$$\text{AVERAGE} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{6-0}\int_0^6 0.032t^4 - 0.26t^3 - 0.478t^2 + 5.82t + 3.8\,dt$$

$$= \frac{1}{6}\left[\frac{0.032t^5}{5} - \frac{0.26t^4}{4} - \frac{0.478t^3}{3} + \frac{5.82t^2}{2} + 3.8t\right]_0^6$$

$$= \frac{1}{6}\left[\left(\frac{0.032}{5}(6)^5 - \frac{0.26}{4}(6)^4 - \frac{0.478}{3}(6)^3 + \frac{5.82}{2}(6)^2 + 3.8(6)\right) - (0)\right]$$

$$= \frac{1}{6}\left[49.7664 - 84.24 - 34.416 + 104.76 + 22.8\right]$$

$$= \frac{1}{6}(58.6704)$$

$$= 9.7784\ \%$$

Question 7. *(5 marks)* Find the area bounded between the function $f(x) = -x^3 + 2x^2 + 3x$ and the x-axis. Sketch a graph to help solve the problem.

$f(x) = -x^3+2x^2+3x \,, \quad g(x) = 0$

INTERSECTION:

$f(x) = g(x)$

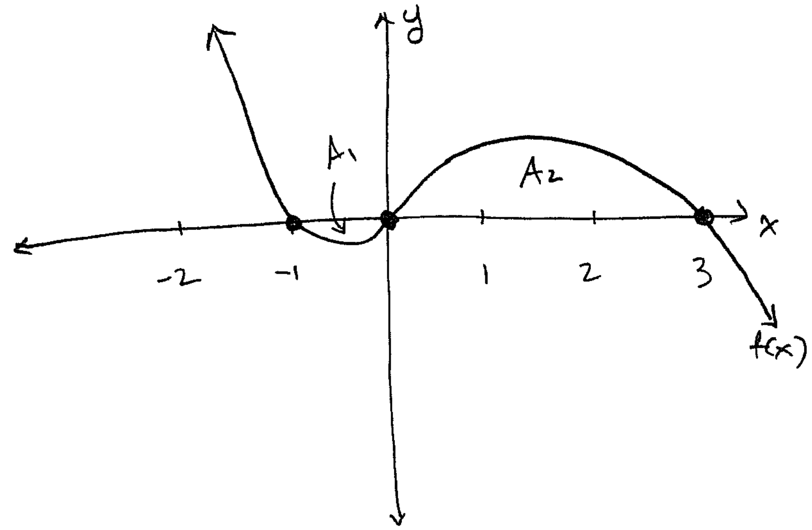
$-x^3+2x^2+3x = 0$

$0 = x^3 - 2x^2 - 3x$

$0 = x(x^2-2x-3)$

$0 = x(x-3)(x+1)$

$\therefore\ x = -1, 0, 3$



$\therefore$ AREA $= A_1 + A_2$

$$= \int_{-1}^{0} [0 - f(x)]\,dx + \int_{0}^{3} [f(x) - 0]\,dx$$

$$= \int_{-1}^{0} -(-x^3+2x^2+3x)\,dx + \int_{0}^{3} (-x^3+2x^2+3x)\,dx$$

$$= \int_{-1}^{0} x^3 - 2x^2 - 3x\,dx + \int_{0}^{3} -x^3+2x^2+3x\,dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3}{2}x^2\right]_{-1}^{0} + \left[-\frac{x^4}{4} + \frac{2}{3}x^3 + \frac{3x^2}{2}\right]_{0}^{3}$$

$$= \left[(0) - \left(\frac{(-1)^4}{4} - \frac{2(-1)^3}{3} - \frac{3}{2}(-1)^2\right)\right] + \left[\left(-\frac{(3)^4}{4} + \frac{2}{3}(3)^3 + \frac{3}{2}(3)^2\right) - (0)\right]$$

$$= -\frac{1}{4} - \frac{2}{3} + \frac{3}{2} - \frac{81}{4} + \frac{54}{3} + \frac{27}{2} = \frac{7}{12} + \frac{45}{4}$$

$$= \frac{71}{6}$$

Bonus Question. *(3 marks)* Given that $g(1) = -1$ and $g(7) = 5$ find

$$\int_1^7 2g'(x)\left(e^{g(x)+1}\right)^3\,dx$$

$$= \int_1^{e^6} 2g'\cancel{(x)}\,(u)^3\,\frac{du}{u\,g'\cancel{(x)}}$$

$$= 2\int_1^{e^6} u^2\,du = 2\,\frac{u^3}{3}\Bigg|_1^{e^6}$$

$$= \frac{2\,(e^6)^3}{3} - \frac{2\,(1)^3}{3}$$

$$= \frac{2}{3}e^{18} - \frac{2}{3}$$

LET $u = e^{g(x)+1}$ CHAIN RULE.

$du = \left(e^{g(x)+1}\right) g'(x)\,dx$

$$\frac{du}{\left(e^{g(x)+1}\right) g'(x)} = dx$$

IF $x = 1$

$\Rightarrow u = e^{g(1)+1}$

$= e^{-1+1}$

$= e^0 = 1$

IF $x = 7$

$\Rightarrow u = e^{g(7)+1}$

$= e^{5+1}$

$= e^6$