Integration by Parts

Recall the product rule for derivatives

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)g'(x) + f'(x)g(x)$$

But this means that

$$\int [f(x)g'(x) + f'(x)g(x)]\,dx = f(x)g(x) + C$$

$$\Rightarrow \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx = f(x)g(x) + C$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

IF WE LET $u = f(x)$ AND $v = g(x)$ THEN WE WOULD GET $du = f'(x)dx$ AND $dv = g'(x)\,dx$ SO THE EQUATION WOULD BE

$$\boxed{\int u\,dv = uv - \int v\,du}$$

THIS IS THE FORMULA FOR INTEGRATION BY PARTS.

TO USE THIS FORMULA WE PICK PART OUR INTEGRAL TO BE $\underline{u}$ AND PART TO BE $\underline{dv}$ AND THEIR PRODUCT SHOULD GIVE US THE ORIGINAL INTEGRAL

Examples: Find the following antiderivatives:

1) $\int x\cos x\,dx = \int u\,dv$

LET

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$= uv - \int v\,du$

$= x\sin x - \int \sin x\,dx$

$= x\sin x - (-\cos x) + C$

$= x\sin x + \cos x + C$

WHEN SELECTING u AND dv

1. WE NEED TO FIND THE ANTIDERIVATIVE OF dV
2. WE WOULD LIKE THE RESULTING INTEGRAL TO BE MORE SIMPLE

LET $u = \cos x \qquad dv = x\,dx$

$du = -\sin x\,dx \qquad v = \frac{x^2}{2}$

$$\int x\cos x\,dx = uv - \int v\,du$$

$$= \underbrace{\frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)\,dx}_{\text{MORE COMPLICATED THAN THE ORIGINAL}}$$

NOT A GOOD CHOICE FOR u AND dV

2) $\int x^2 e^x\,dx = \int u\,dv$

LET

$u = x^2 \qquad dv = e^x\,dx$

$du = 2x\,dx \qquad v = e^x$

$= uv - \int v\,du$

$= x^2 e^x - \int e^x (2x)\,dx$

$= x^2 e^x - 2\int x e^x\,dx$

LET

$U = x \qquad dV = e^x\,dx$

$dU = dx \qquad V = e^x$

$= x^2 e^x - 2\left[UV - \int V\,dU\right]$

$= x^2 e^x - 2\left[x e^x - \int e^x\,dx\right]$

$= x^2 e^x - 2x e^x + 2\int e^x\,dx$

$= x^2 e^x - 2x e^x + 2e^x + C$

LET

$u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x}dx$ $\quad v = x$

3) $\int \ln x \, dx$

$= uv - \int v\,du$

$= (\ln x)(x) - \int x \cdot \frac{1}{x}\,dx$

$= x\ln x - \int dx$

$= x\ln x - x + C$

CHECK:

$\frac{d}{dx}\left[x\ln x - x + C\right] = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 + 0$

$= \ln x + 1 - 1$

$= \ln x$

LET

$u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

TRY $\int x \ln x \, dx$

$= uv - \int v\,du$

$= (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right) \cdot \frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C$

$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C$

4) $\int \frac{xe^x}{(x+1)^2}\,dx$

LET

$u = xe^x \qquad dv = \frac{1}{(x+1)^2}\,dx$

$du = (e^x + xe^x)\,dx$

$v = \int \frac{1}{(x+1)^2}\,dx = \int (x+1)^{-2}\,dx$

$= \int t^{-2}\,dt$

LM $t = x+1$

$dt = dx$

$= \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C$

$= -\frac{1}{x+1} + \cancel{C}$

$= uv - \int v\,du$

$= xe^x\left(-\frac{1}{x+1}\right) - \int \left(-\frac{1}{x+1}\right)(e^x + xe^x)\,dx$

$= -\frac{xe^x}{x+1} + \int \frac{e^x + xe^x}{x+1}\,dx$

$= \frac{-xe^x}{x+1} + \int \frac{e^x\,\cancel{(1+x)}}{\cancel{x+1}}\,dx$

$= -\frac{xe^x}{x+1} + \int e^x\,dx = -\frac{xe^x}{x+1} + e^x + C$

5) $\underbrace{\int e^x \cos x\, dx}_{I}$

$= uv - \int v\,du$

$= e^x \sin x - \int (\sin x)(e^x)\,dx$

$= e^x \sin x - \left[UV - \int V\,dU \right]$

$= e^x \sin x - \left[e^x(-\cos x) - \int (-\cos x) e^x\,dx \right]$

$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$

$I = e^x \sin x + e^x \cos x - I$

$2I = e^x \sin x + e^x \cos x + C_1$

$I = \dfrac{e^x \sin x + e^x \cos x}{2} + C$

Let

$u = e^x \qquad dv = \cos x\,dx$

$du = e^x\,dx \qquad v = \sin x$

Let

$U = e^x \qquad dV = \sin x\,dx$

$dU = e^x dx \qquad V = -\cos x$

Integration by parts for a definite integral:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example: Evaluate the following definite integrals

1) $\int_1^e x^3(\ln x)^2\,dx$

LET

$u = (\ln x)^2 \qquad dv = x^3 dx$

$du = 2\ln x \cdot \frac{1}{x}dx \qquad v = \frac{x^4}{4}$

$$= uv\Big|_1^e - \int_1^e v\,du$$

$$= (\ln x)^2\left(\frac{x^4}{4}\right)\Big|_1^e - \int_1^e \left(\frac{x^4}{4}\right)2\ln x \cdot \frac{1}{x}dx$$

$$= \left[(\overset{1}{\cancel{\ln e}})^2\frac{e^4}{4} - (\overset{0}{\cancel{\ln 1}})^2\frac{1^4}{4}\right] - \frac{1}{2}\int_1^e x^3\ln x\,dx$$

$$= \frac{e^4}{4} - \frac{1}{2}\int_1^e x^3\ln x\,dx$$

LET

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^4}{4}$

$$= \frac{e^4}{4} - \frac{1}{2}\left[uv\Big|_1^e - \int_1^e v\,du\right]$$

$$= \frac{e^4}{4} - \frac{1}{2}\left[\ln x \cdot \frac{x^4}{4}\Big|_1^e - \int_1^e \frac{x^4}{4}\cdot\frac{1}{x}dx\right]$$

$$= \frac{e^4}{4} - \frac{1}{2}\left[(\ln e)\frac{e^4}{4} - (\ln 1)\cdot\frac{1^4}{4}\right] + \frac{1}{2}\int_1^e \frac{x^3}{4}dx$$

$$= \frac{e^4}{4} - \frac{1}{2}\left[\frac{e^4}{4} - 0\right] + \frac{1}{8}\int_1^e x^3\,dx = \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\left[\frac{x^4}{4}\right]_1^e$$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\left[\frac{e^4}{4} - \frac{1}{4}\right] = \frac{e^4}{4} - \frac{e^4}{8} + \frac{e^4}{32} - \frac{1}{32} = \frac{5e^4}{32} - \frac{1}{32}$$

2) $\int_0^1 (x-3)e^{3x}\,dx$

Let
$u = x-3 \qquad dv = e^{3x}dx$
$du = dx \qquad v = \frac{e^{3x}}{3}$

$= uv\Big|_0^1 - \int_0^1 v\,du$

$= (x-3)\frac{e^{3x}}{3}\Big|_0^1 - \int_0^1 \frac{e^{3x}}{3}\,dx$

$= \left[(1-3)\frac{e^3}{3} - (0-3)\frac{e^0}{3}\right] - \frac{1}{3}\int_0^1 e^{3x}\,dx$

$= -\frac{2}{3}e^3 + 1 - \frac{1}{3}\left[\frac{e^{3x}}{3}\right]_0^1$

$= -\frac{2}{3}e^3 + 1 - \frac{1}{3}\left[\frac{e^3}{3} - \frac{e^0}{3}\right]$

$= -\frac{2}{3}e^3 + 1 - \frac{1}{9}e^3 + \frac{1}{9}$

$= -\frac{7}{9}e^3 + \frac{10}{9}$

Example (page 489 #41): Sales of the latest recording by Brittania, a British rock group are currently $2te^{-0.1t}$ per week (each unit representing 10 000 CDs), where t denotes the number of weeks since the recording's release. Find an expression that gives the total number of CDs sold as a function of t.

$f(t) = 2te^{-0.1t}$, $f(2) = 2(2)e^{-0.1(2)} = 3.2749$

$\therefore$ 2 WEEKS AFTER THE CD IS RELEASED IT IS SELLING AT A RATE OF 32 749 CDS PER WEEK.

LET $g(t)$ BE THE FUNCTION THAT GIVES THE TOTAL NUMBER OF CDS SOLD BY WEEK t.

$f(t)$ GIVES US A RATE (HOW FAST CDS ARE BEING SOLD)

$f(t) = g'(t) \Rightarrow \int f(t)\,dt = g(t)$

$g(t) = \int f(t)\,dt = \int 2te^{-0.1t}\,dt$

LET $u = 2t$ $\quad dv = e^{-0.1t}\,dt$

$du = 2\,dt$ $\quad v = \dfrac{e^{-0.1t}}{-0.1}$

$= -2t(10e^{-0.1t}) - \int (-10e^{-0.1t})\,2\,dt$

$= -20te^{-0.1t} + 20\int e^{-0.1t}\,dt$

$= -20te^{-0.1t} + 20\left[\dfrac{e^{-0.1t}}{-0.1}\right] + C = -20te^{-0.1t} - 200e^{-0.1t} + C$

HOW DO WE FIND C? WE KNOW $g(0) = 0$

$0 = g(0) = -20(0)e^{-0.1(0)} - 200e^{0} + C$

$0 = 0 - 200 + C \Rightarrow 200 = C$

$\therefore g(t) = -20te^{-0.1t} - 200e^{-0.1t} + 200$