

## Partial Fractions

In this section we will examine a method of integration for rational functions.

Rational Function:  $\frac{p(x)}{q(x)}$  where  $p(x)$  and  $q(x)$  are polynomials.

### **Common Types of Integrals**

We will need to be able to integrate the following integrals the following integrals.

$$\mathbf{1)} \int \frac{1}{(ax+b)^n} dx$$

For this type of integral we use a substitution.

Examples: Find the following antiderivatives.

$$\mathbf{a)} \int \frac{1}{(3x+2)} dx$$

$$\mathbf{b)} \int \frac{4}{(5x-7)^3} dx$$

$$2) \int \frac{ax+b}{(cx^2+dx+g)^n} dx$$

We want to try to make the numerator look like the derivative of the denominator so we can use a substitution.

Examples: Find the following antiderivatives.

$$\mathbf{a)} \int \frac{8x+3}{4x^2+3x-4} dx$$

$$\mathbf{b)} \int \frac{2x+1}{(2x^2+2x+5)^3} dx$$

$$\mathbf{c)} \int \frac{5x-7}{2x^2+2} dx$$

Remember: the goal of this section is to integrate rational functions,  $p(x)/q(x)$ . If the degree of the numerator  $p(x)$  is **greater than** or **equal to** the degree of the denominator  $q(x)$  we first perform **long division**.

Example: Divide the following using long division.

a) 
$$\frac{3x^2 + 4x - 3}{x + 2}$$

b) 
$$\frac{3x^3 - 5x + 2}{x - 1}$$

**c)**  $\frac{4x^3 + 8x^2 - x + 6}{x^2 + 1}$

Example: Find the following.

**1)**  $\int \frac{x^3 + 3x^2}{x^2 + 1} dx$

**2)**  $\int \frac{x^3 + x}{x - 1} dx$

Partial Fractions: The next step is to factor the denominator as much as possible. It can be shown that any polynomial  $q(x)$  can be factored as a product of linear factors (of the form  $ax + b$ ) and irreducible quadratic factors (of the form  $ax^2 + bx + c$  where  $b^2 - 4ac < 0$ ).

Important: All remaining quadratics must be irreducible!!!

Once  $q(x)$  is factored the next step is to express the function as a sum of partial fractions of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

To illustrate, recall that we can combine fractions

$$\frac{5}{x + 3} - \frac{2}{x + 1}$$

and so we can separate

**Case 1:** The denominator  $q(x)$  is a product of distinct linear factors  $ax + b$ .

$$q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$$

For each linear factor we need one term of the type  $\frac{A}{ax+b}$ .

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}$$

Example: Write the following rational function as a sum or simple fractions

$$\frac{x+5}{(x-4)(x-1)}$$



Example: Find

$$1) \int \frac{x+5}{x^2-5x+4} dx$$

$$2) \int \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$$

$$3) \int \frac{2x-2}{(x+5)(x+2)(x-3)} dx$$

**Case 2:** The denominator  $q(x)$  is a product of linear factors  $ax + b$  some of which are repeated. For each  $(ax + b)^n$  we need

$$\frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_n}{(ax + b)^n}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2 + 2x - 3}{x^2(2x + 1)^3}$$

Example: Find

$$1) \int \frac{x^2 - 2x - 5}{x^3 - 5x^2} dx$$



$$2) \int \frac{5x^2 - 9x}{(x-4)(x-1)^2} dx$$

$$3) \int \frac{5x^2 + 3x - 2}{x^3 + 2x^2} dx$$

**Case 3:** The denominator  $q(x)$  contains irreducible quadratic factors  $ax^2 + bx + c$ , where  $b^2 - 4ac < 0$ , none of which are repeated. For each  $ax^2 + bx + c$  we need a factor of the type

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2 + 2x}{(x + 1)^3(x^2 + x + 1)(3x^2 + 2)}$$

Example: Find

1)  $\int \frac{x-3}{(x+2)(x^2+6)} dx$



$$2) \int \frac{-5x^3 + 2x^2 + x + 3}{x^4 + x^2} dx$$