Partial Fractions

In this section we will examine a method of integration for ~~frational~~ rational functions.

Rational Function: $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Ex: $\int \frac{x^4-3x^3+2x-1}{x^2-3x+2}\,dx$

Common Types of Integrals

We will need to be able to integrate the following integrals the following integrals.

1) $\displaystyle\int \frac{1}{(ax+b)^n}\,dx$

For this type of integral we use a substitution.

Examples: Find the following antiderivatives.

a) $\displaystyle\int \frac{1}{(3x+2)}\,dx = \frac{\ln|3x+2|}{3} + C$

$u = 3x+2$

$du = 3dx$

b) $\displaystyle\int \frac{4}{(5x-7)^3}\,dx = 4\int \frac{1}{(5x-7)^3}\,dx = 4\int u^{-3}\,\frac{du}{5}$

$u = 5x-7$

$du = 5dx$

$= \dfrac{4}{5}\,\dfrac{u^{-2}}{-2} + C = -\dfrac{4}{10}\cdot\dfrac{1}{(5x-7)^2} + C$

2) $\displaystyle\int \frac{ax+b}{(cx^2+dx+g)^n}\,dx$

We want to try to make the numberator look like the derivative of the denominator so we can use a substitution.

Examples: Find the following antiderivatives.

a) $\displaystyle\int \frac{8x+3}{4x^2+3x-4}\,dx$

LET $u = 4x^2+3x-4$

$du = (8x+3)dx$

$\dfrac{du}{8x+3} = dx$

$$= \int \frac{8x+3}{u} \cdot \frac{du}{8x+3} = \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln|4x^2+3x-4| + C$$

b) $\displaystyle\int \frac{2x+1}{(2x^2+2x+5)^3}\,dx$

LET $u = 2x^2+2x+5$

$du = (4x+2)dx$

$\dfrac{du}{2(2x+1)} = dx$

$$= \int \frac{\cancel{2x+1}}{u^3}\,\frac{du}{2(\cancel{2x+1})} = \frac{1}{2}\int u^{-3}\,du$$

$$= \frac{1}{2}\frac{u^{-2}}{-2} + C = -\frac{1}{4}\cdot\frac{1}{(2x^2+2x+5)^2} + C$$

c) $\displaystyle\int \frac{5x-7}{2x^2+2}\,dx = \int \frac{5x}{2x^2+2}\,dx - \int \frac{7}{2x^2+2}\,dx$

LET $u = 2x^2+2$

$du = 4x\,dx$

$\dfrac{du}{4x} = dx$

$$= \int \frac{5x}{u}\cdot\frac{du}{4x} - \int \frac{7}{2x^2+2}\,dx = \frac{5}{4}\ln|2x^2+2| - \int \frac{7}{2x^2+2}\,dx$$

$$= \frac{5}{4}\ln|2x^2+2| - \frac{7}{2}\int \frac{1}{x^2+1}\,dx = \frac{5}{4}\ln|2x^2+2| - \frac{7}{2}\arctan x + C$$

Remeber: the goal of this section is to integrate rational functions, $p(x)/q(x)$. If the degree of the numberator $p(x)$ is **greater than** or **equal to** the degree of the denominator $q(x)$ we first perform **long division**.
Example: Divide the following using long division.

a) $\dfrac{3x^2+4x-3}{x+2}$

$$\begin{array}{rl} & 3x - 2 \\ x+2 \,\big) & 3x^2 + 4x - 3 \\ 3x(x+2) \to & -(3x^2 + 6x) \quad \downarrow \\ & -2x - 3 \\ -2(x+2) \to & -(-2x - 4) \\ & 1 \leftarrow \text{REMAINDER} \end{array}$$

(DEGREE IS LESS THAN THE DENOMINATOR)

$$\frac{17}{7} = 2 + \frac{3}{7} \leftarrow \text{REMAINDER}$$

$$\frac{11}{2} = 5 + \frac{1}{2}$$

$$\frac{3x^2+4x-3}{x+2} = 3x-2 + \frac{1}{x+2}$$

b) $\dfrac{3x^3-5x+2}{x-1}$

$$\begin{array}{rl} & 3x^2 + 3x - 2 \\ x-1 \,\big) & 3x^3 + 0x^2 - 5x + 2 \\ & -(3x^3 - 3x^2) \quad \downarrow \\ & 3x^2 - 5x \quad \downarrow \\ & -(3x^2 - 3x) \\ & -2x + 2 \\ & -(-2x + 2) \\ & 0 \end{array}$$

$$\frac{3x^3-5x+2}{x-1} = 3x^2+3x-2 + \frac{0}{x-1}$$

$$= 3x^2+3x-2$$

c) $\dfrac{4x^3+8x^2-x+6}{x^2+1}$

$$
\begin{array}{r l}
 & 4x + 8 \\
\cline{2-2}
x^2+0x+1 & \big) \; 4x^3 + 8x^2 - x + 6 \\
 & -(4x^3 + 0x^2 + 4x) \\
\cline{2-2}
 & \qquad 8x^2 - 5x + 6 \\
 & \quad -(8x^2 + 0x + 8) \\
\cline{2-2}
 & \qquad\qquad -5x - 2
\end{array}
$$

$$\frac{4x^3+8x^2-x+6}{x^2+1} = 4x+8 - \frac{5x+2}{x^2+1}$$

Example: Find the following.

1) $\displaystyle\int \frac{x^3+3x^2}{x^2+1}\,dx = I$

$$
\begin{array}{r|l}
 & x \quad + 3 \\
\hline
x^2+0x+1 & x^3 + 3x^2 + 0x + 0 \\
 & -(x^3 + 0x^2 + x) \\
\hline
 & 3x^2 - x + 0 \\
 & -(3x^2 + 0x + 3) \\
\hline
 & -x - 3
\end{array}
$$

$$I = \int x+3 - \frac{x+3}{x^2+1}\,dx = \int x+3\,dx - \int \underbrace{\frac{x}{x^2+1}}_{\substack{u = x^2+1 \\ du = 2x\,dx}} dx - \int \frac{3}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + 3x - \int \frac{\cancel{x}}{u}\,\frac{du}{2\cancel{x}} - 3\int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + 3x - \frac{1}{2}\ln|u| - 3\arctan x + C$$

$$= \frac{x^2}{2} + 3x - \frac{1}{2}\ln|x^2+1| - 3\arctan x + C$$

2) $\int \frac{x^3+x}{x-1}\,dx = I$

$$
\begin{array}{r l}
 & x^2 + x + 2 \\
\cline{2-2}
x-1 \,\big) & x^3 + 0x^2 + x + 0 \\
 & -(x^3 - x^2) \\
\cline{2-2}
 & \qquad x^2 + x \\
 & \qquad -(x^2 - x) \\
\cline{2-2}
 & \qquad\qquad 2x + 0 \\
 & \qquad\qquad -(2x - 2) \\
\cline{2-2}
 & \qquad\qquad\qquad 2
\end{array}
$$

$$I = \int x^2 + x + 2 + \frac{2}{x-1}\,dx = \int x^2 + x + 2\,dx + 2\int \frac{1}{x-1}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

ONCE THE DEGREE OF THE NUMERATOR IS LESS THAN THE DENOMINATOR

Partial Fractions: The next step ✓ is to factor the denominator as much as possible. It can be shown that any polynomial $q(x)$ can be factored as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form ax^2+bx+c where $b^2-4ac<0$).

$q(x)$

FACTOR

FACTORS OF THE FORM $ax+b$

FACTORS OF THE FORM ax^2+bx+c WHERE $b^2-4ac<0$

Important: All remaining quadratics must be irreducible!!!

Once $q(x)$ is factored the next step is to express the function as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

To illustrate, recall that we can combine fractions

$$\frac{5}{x+3}-\frac{2}{x+1} = \frac{5(x+1)}{(x+3)(x+1)} - \frac{2(x+3)}{(x+3)(x+1)}$$

$$= \frac{(5x+5)-(2x+6)}{(x+3)(x+1)} = \frac{3x-1}{x^2+4x+3}$$

and so we can separate

$$\frac{3x-1}{x^2+4x+3} = \frac{5}{x+3} - \frac{2}{x+1}$$

HOW DO WE DO THIS?

THERE ARE FOUR CASES THAT WE WILL LOOK AT

Case 1: The denominator $q(x)$ is a product of distinct linear factors $ax+b$.

$$q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$$

Ex: $q(x) = (2x+1)(3x-7)(5x+2)\ldots(6x-1)$

For each linear factor we need one term of the type $\frac{A}{ax+b}$.

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_n}{a_nx+b_n}$$

Example: Write the following rational function as a sum or simple fractions

$$\frac{x+5}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$$

← HOW DO WE FIND THESE CONSTANTS?

MULTIPLY BOTH SIDES BY $(x-4)(x-1)$.

THIS EQUALITY IS TRUE FOR ALL x VALUES.

$$\cancel{(x-4)(x-1)} \cdot \frac{x+5}{\cancel{(x-4)(x-1)}} = (\cancel{x-4})(x-1)\frac{A}{\cancel{x-4}} + (x-4)(\cancel{x-1})\frac{B}{\cancel{x-1}}$$

$$x+5 = (x-1)A + (x-4)B$$

IF $\underline{x=1}$

$$1+5 = (1-1)A + (1-4)B$$

$$6 = 0A - 3B$$

$$\underline{-2 = B}$$

$\underline{\text{IF } x=4}$

$$4+5 = (4-1)A + (4-4)(-2)$$

$$9 = 3A + 0$$

$$3 = A$$

$$\therefore \frac{x+5}{(x-4)(x-1)} = \frac{3}{x-4} + \frac{(-2)}{x-1}$$

Example: Find

1) $\int \frac{x+5}{x^2-5x+4}\,dx = \int \frac{x+5}{(x-4)(x-1)}\,dx$

$= \int \frac{3}{x-4} - \frac{2}{x-1}\,dx = 3\int \frac{1}{x-4}\,dx - 2\int \frac{1}{x-1}\,dx$

$= 3\ln|x-4| - 2\ln|x-1| + C$

2) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

3) $\displaystyle\int \frac{2x-2}{(x+5)(x+2)(x-3)}\,dx$

Case 2: The denominator $q(x)$ is a product of linear factors $ax+b$ some of which are repeated. For each $(ax+b)^n$ we need

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \ldots + \frac{A_n}{(ax+b)^n}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2+2x-3}{x^2(2x+1)^3}$$

Example: Find

1) $\displaystyle\int \frac{x^2-2x-5}{x^3-5x^2}\,dx$

$$\frac{-12}{5} = -4A$$

$$\frac{3}{5} = A$$

$$I = \int \frac{\left(\frac{3}{5}\right)}{x} + \frac{1}{x^2} + \frac{\left(2/5\right)}{x-5} \, dx$$

$$= \frac{3}{5}\int \frac{1}{x}\, dx + \int x^{-2}\, dx + \frac{2}{5}\int \frac{1}{x-5}\, dx$$

$$= \frac{3}{5}\ln|x| - x^{-1} + \frac{2}{5}\ln|x-5| + C$$

2) $\int \frac{5x^2-9x}{(x-4)(x-1)^2}\,dx = I$

$$\frac{5x^2-9x}{(x-4)(x-1)^2} = \frac{A}{x-4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$5x^2-9x = A(x-1)^2 + B(x-1)(x-4) + C(x-4)$$

IF $x=1$

$5-9 = A(0) + B(0) + C(-3)$

$-4 = -3C \Rightarrow C = 4/3$

IF $x=4$

$80-36 = A(9) + B(0) + C(0)$

$44 = 9A \rightarrow A = \frac{44}{9}$

IF $x=0$

$$0 = \frac{44}{9} + B(-1)(-4) + \frac{4}{3}(-4)$$

$$0 = \frac{44}{9} + 4B - \frac{16}{3} \Rightarrow \frac{16}{3} - \frac{44}{9} = 4B \Rightarrow \frac{4}{9} = 4B \Rightarrow \frac{1}{9} = B$$

$$I = \int \frac{(44/9)}{x-4} + \frac{(1/9)}{x-1} + \frac{(4/3)}{(x-1)^2}\,dx$$

$$= \frac{44}{9}\int \frac{1}{x-4}\,dx + \frac{1}{9}\int \frac{1}{x-1}\,dx + \frac{4}{3}\int \underbrace{(x-1)}_{u=x-1 \Rightarrow du=dx}{}^{-2}\,dx$$

$$= \frac{44}{9}\ln|x-4| + \frac{1}{9}\ln|x-1| + \frac{4}{3}\frac{(x-1)^{-1}}{-1} + C$$

$$= \frac{44}{9}\ln|x-4| + \frac{1}{9}\ln|x-1 - \frac{4}{3}\cdot\frac{1}{x-1} + C$$

3) $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = I$ FACTOR: $x^3+2x^2 = x^2(x+2)$

$$\frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$$

IF $x=0$

$-2 = A(0) + B(2) + C(0)$

$-1 = B$

IF $x=-2$

$20-6-2 = A(0)+B(0)+C(4)$

$12 = 4C \Rightarrow C = 3$

IF $x=1$

$5+3-2 = A(1)(3) + (-1)(3) + (1)(3)$

$6 = 3A \Rightarrow 2 = A$

$$\therefore I = \int \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\,dx$$

$$= 2\int \frac{1}{x}dx - \int \frac{1}{x^2}dx + 3\int \frac{1}{x+2}dx$$

$$= 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C$$

Case 3: The denominator $q(x)$ contains irreducible quadratic factors ax^2+bx+c, where $b^2-4ac<0$, none of which are repeated. For each ax^2+bx+c we need a factor of the type

$$\frac{Ax+B}{ax^2+bx+c}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2+2x}{(x+1)^3\underbrace{(x^2+x+1)}_{(1)^2-4(1)(1)<0}(3x^2+2)} \longleftarrow \quad 0^2-4(3)(2)<0$$

$$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+x+1} + \frac{Fx+G}{3x^2+2}$$

Example: Find

1) $\cancel{\displaystyle\int \frac{4x^2+9x+2}{x^3+3x^2+x}\,dx}$

FACTOR; $x^3+3x^2+x = x(x^2+3x+1)$

$(3)^2 - 4($

$$\int \frac{x-3}{(x+2)(x^2+6)}\,dx$$

$$\frac{x-3}{(x+2)(x^2+6)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+6}$$

$$x-3 = A(x^2+6) + (Bx+C)(x+2)$$

$$= \underline{Ax^2} + 6A + \underline{Bx^2} + \overline{Cx} + \overline{2Bx} + 2C$$

$$= Ax^2 + Bx^2 + Cx + 2Bx + 6A + 2C$$

$$= (A+B)x^2 + (C+2B)x + (6A+2C)$$

$$\therefore \left.\begin{aligned} 0 &= A+B \\ 1 &= C+2B \\ -3 &= 6A+2C \end{aligned}\right\}$$

$A = -B \Rightarrow -A = B$

$\Rightarrow 1 = C + 2(-A) \Rightarrow 1+2A = C$

$-3 = 6A + 2(1+2A)$

$-3 = 6A + 2 + 4A$

$-5 = 10A$

$-1/2 = A$

$\therefore B = 1/2 \Rightarrow 1 = C + 2(1/2) \Rightarrow 0 = C$

2) $\displaystyle\int \frac{-5x^3+2x^2+x+3}{x^4+x^2}\,dx = I$ FACTOR: $x^4+x^2 = x^2(x^2+1)$

$0^2-4(1)(1)<0$

IRREDUCIBLE

$$\frac{-5x^3+2x^2+x+3}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$$

$$-5x^3+2x^2+x+3 = Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$$

~~IF $x=0$~~

~~$3 = A(0)+B(1)+(C(0)+D)(0)$~~

~~$3=B$~~

NEW METHOD:

$$-5x^3+2x^2+x+3 = Ax^3+Ax+Bx^2+B+Cx^3+Dx^2$$

$$-5x^3+2x^2+x+3 = Ax^3+Cx^3+Bx^2+Dx^2+Ax+B$$
$$= (A+C)x^3+(B+D)x^2+Ax+B$$

$\therefore$ $-5 = A+C$ $\quad 1=A$

$2 = B+D$ $\quad 3=B$

$-5 = 1+C \Rightarrow -6=C$

$2 = 3+D \Rightarrow -1 = D$

$$I = \int \frac{1}{x}+\frac{3}{x^2}+\frac{-6x+(-1)}{x^2+1}\,dx$$

$$= \int \frac{1}{x}dx + 3\int\frac{1}{x^2}dx - 6\int\frac{x}{x^2+1}dx - \int\frac{1}{x^2+1}dx$$

$u=x^2+1 \Rightarrow du = 2x\,dx$

$$= \ln|x| + 3\frac{x^{-1}}{-1} - 6\int\frac{x}{u}\frac{du}{2x} - \arctan x$$

$$= \ln|x| - \frac{3}{x} - 3\ln|x^2+1| - \arctan x + C$$