Partial Fractions Part 2

<u>Case 4</u>: The denominator $q(x)$ contains a repeated irreducible quadratic factor ax^2+bx+c, where $b^2-4ac<0$. For each $(ax^2+bx+c)^n$ in the denominator we need

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{3x^4+2x-1}{x(3x-2)^3(x^2+x+1)(x^2+1)^2}$$

$$=\frac{A}{x}+\frac{B}{3x-2}+\frac{C}{(3x-2)^2}+\frac{D}{(3x-2)^3}+\frac{Ex+F}{x^2+x+1}+\frac{Gx+H}{x^2+1}+\frac{Ix+J}{(x^2+1)^2}$$

.

Example: Find

1) $I = \displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$ $\quad 0^2-4(1)(1) < 0$ IRREDUCIBLE

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\begin{aligned}1-x+2x^2-x^3 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2+Ex \\ &= \underline{Ax^4} + 2Ax^2 + A + \underline{Bx^4} + Cx^3 + Bx^2 + Cx + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (B+D+2A)x^2 + (E+C)x + A\end{aligned}$$

$\therefore$ $\boxed{1 = A}$ $\qquad 2 = B+D+2A \qquad 0 = A+B$

$-1 = E+C \quad \boxed{-1 = C}$

$-1 = E - 1$

$\Rightarrow \boxed{E = 0}$

$0 = 1 + B$

$\boxed{-1 = B}$

$2 = (-1) + D + 2(1)$

$2 = D+1 \Rightarrow \boxed{D = 1}$

$$I = \int \frac{1}{x} + \frac{(-1)x + (-1)}{x^2+1} + \frac{(1)x + 0}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

LET $u = x^2+1$

$du = 2x\,dx$

$$= \ln|x| - \int \frac{x}{u}\frac{du}{2x} - \arctan x + \int \frac{x}{u^2}\frac{du}{2x}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \arctan x + \frac{1}{2}\frac{(x^2+1)^{-1}}{-1} + C$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \arctan x - \frac{1}{2}\cdot\frac{1}{x^2+1} + C$$

2) $$\int \frac{3x^3+9x^2+16x+10}{(x^2+2x+3)^2}\,dx$$

$$\frac{3x^3+9x^2+16x+10}{(x^2+2x+3)^2} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2}$$

$$3x^3+9x^2+16x+10 = (Ax+B)(x^2+2x+3) + Cx+D$$

$$3x^3+9x^2+16x+10 = Ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx+D$$

$$= Ax^3+(2A+B)x^2+(3A+2B+C)x+(3B+D)$$

$\therefore 3 = A$, $9 = 2A+B$, $16 = 3A+2B+C$, $10 = 3B+D$

$\therefore 9 = 2(3)+B$; $3 = B$

$\therefore 16 = 3(3)+2(3)+C$; $16-9-6 = C$; $1 = C$

$10 = 3(3)+D$; $1 = D$

$$\therefore I = \int \frac{3x+3}{x^2+2x+3} + \frac{x+1}{(x^2+2x+3)^2}\,dx$$

$$= \int \frac{3x+3}{x^2+2x+3}\,dx + \int \frac{x+1}{(x^2+2x+3)^2}\,dx$$

$$\boxed{\begin{aligned} &\text{LET } u = x^2+2x+3 \\ &du = (2x+2)\,dx \\ &\frac{du}{2(x+1)} = dx \end{aligned}}$$

$$= \int \frac{3\cancel{(x+1)}}{u}\,\frac{du}{2\cancel{(x+1)}} + \int \frac{\cancel{x+1}}{u^2}\,\frac{du}{2\cancel{(x+1)}}$$

$$= \frac{3}{2}\int \frac{1}{u}\,du + \frac{1}{2}\int u^{-2}\,du = \frac{3}{2}\ln|x^2+2x+3| + \frac{1}{2}\frac{(x^2+2x+3)^{-1}}{-1} + C$$

$$= \frac{3}{2}\ln|x^2+2x+3| - \frac{1}{2}\cdot\frac{1}{x^2+2x+3} + C$$

Partial fraction examples using mixed cases: Find

1) $\int \frac{2x^3+2x^2-95x+40}{x^3+x^2-20x}\,dx$

$\frac{5}{3} = 1 + \frac{2}{3}$

$$\begin{array}{r} 2 \\ x^3+x^2-20x \,\big)\, 2x^3+2x^2-95x+40 \\ -(2x^3+2x^2-40x) \\ \hline -55x+40 \end{array}$$

$$\therefore \frac{2x^3+2x^2-95x+40}{x^3+x^2-20x} = 2 + \frac{-55x+40}{\underbrace{x^3+x^2-20x}_{?}}$$

FACTOR: $x^3+x^2-20x = x(x^2+x-20) = x(x+5)(x-4)$

$$\frac{-55x+40}{x(x+5)(x-4)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-4}$$

$$-55x+40 = A(x+5)(x-4) + Bx(x-4) + Cx(x+5)$$

IF $x = -5$

$275+40 = A(0) + B(-5)(-9) + C(0)$

$315 = 45B \Rightarrow 7 = B$

IF $x=4$

$-220+40 = A(0) + B(0) + C(4)(9)$

$-180 = 36C \Rightarrow C = -5$

IF $x = 0$

$40 = A(5)(-4)$

$-2 = A$

$$\therefore I = \int 2 - \frac{2}{x} - \frac{7}{x+5} - \frac{5}{x-4}\,dx$$

$$= 2x - 2\ln|x| - 7\ln|x+5| - 5\ln|x-4| + C$$

2) $\int \frac{x+1}{(x^2+2x+10)^2}\,dx$

$2^2-4(10)(1)<0$ IRREDUCIBLE

$$\frac{x+1}{(x^2+2x+10)^2} = \frac{Ax+B}{x^2+2x+10} + \frac{Cx+D}{(x^2+2x+10)^2}$$

$$\int \frac{5}{x+3}\,dx$$

$A=0,\ B=0,\ C=1,\ D=1$

$$\int \frac{x+1}{(x^2+2x+10)^2}\,dx = \int \frac{\cancel{x+1}}{u^2}\,\frac{du}{2(\cancel{x+1})}$$

$$= \frac{1}{2}\int u^{-2}\,du = \frac{1}{2}\cdot\frac{(x^2+2x+10)^{-1}}{-1} + C$$

$$= -\frac{1}{2}\cdot\frac{1}{x^2+2x+10} + C$$

LET

$u = x^2+2x+10$

$du = 2x+2\,dx$

$\frac{du}{2(x+1)} = dx$

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3) $\displaystyle\int \frac{3x^4+5x^3+2x+3}{(2x+1)(x^2+1)^2}\,dx \neq \frac{A}{2x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

IRREDUCIBLE

$$\frac{3x^4+5x^3+2x+3}{(2x+1)(x^2+1)^2} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$3x^4+5x^3+2x+3 = A(x^2+1)^2 + (Bx+C)(2x+1)(x^2+1) + (Dx+E)(2x+1)$$

$$= A(x^4+2x^2+1) + (Bx+C)(2x^3+x^2+2x+1) + 2Dx^2+2Ex+Dx+E$$

$$= Ax^4 + 2Ax^2 + A + 2Bx^4 + Bx^3 + 2Bx^2 + Bx$$
$$+ 2Cx^3 + Cx^2 + 2Cx + C + 2Dx^2 + 2Ex + Dx + E$$

$$= (A+2B)x^4 + (B+2C)x^3 + (2A+2B+C+2D)x^2$$
$$+ (B+2C+2E+D)x + (A+C+E)$$

$3 = A+2B$, $5 = B+2C$, $0 = 2A+2B+C+2D$, $2 = B+2C+2E+D$

$3 = A+C+E$

$\frac{3-A}{2} = B \rightarrow 5 = \frac{3-A}{2} + 2C$

$\therefore \frac{3-1}{2} = 1 = B$

$5 - \frac{3}{2} + \frac{A}{2} = 2C$

$\frac{7}{2} + \frac{A}{2} = 2C$

$\frac{7}{4} + \frac{A}{4} = C$

$3 - A - \frac{7}{4} - \frac{A}{4} = E$

$\frac{5}{4} - \frac{5A}{4} = E$

$0 = 2A + 2\left(\frac{3-A}{2}\right) + \left(\frac{7}{4}+\frac{A}{4}\right) + 2D$

$0 = 2A + 3 - A + \frac{7}{4} + \frac{A}{4} + 2D$

$\cancel{3 = A + C + E}$

$\cancel{= A + \frac{7}{4} + \frac{A}{4} + \frac{5}{4} - \frac{5A}{4}}$

$0 = \frac{5A}{4} + \frac{19}{4} + 2D$

$-\frac{5A}{4} - \frac{19}{4} = 2D \Rightarrow \cancel{-\frac{5A}{8} - \frac{19}{8} = D}$

$-\frac{5A}{2}$ $-\frac{5A}{8}$

$2 = B + 2C + 2E + D$

$2 = \frac{3-A}{2} + 2\left(\frac{7}{4}+\frac{A}{4}\right) + 2\left(\frac{5}{4} - \frac{5A}{4}\right) + \left(-\frac{5A}{8} - \frac{19}{8}\right) = \frac{3}{2} + \frac{7}{2} + \frac{5}{2} - \frac{19}{8} - \cancel{\frac{A}{2}} + \cancel{\frac{A}{2}}$