# The Substitution Rule

Suppose we wanted to find the antiderivative

$$\int 3x^2\sqrt{1+x^3}dx.$$

Unfortunately none of the integration formulas we have seen so far tell us how to do this. Indeed, there is no product rule for antiderivatives in the same way that there is for derivatives.

We will want to use a change in variables in this case. First we will examine the method and then we will look at why it works.

If we let $u = 1+x^3$ then the differential of $u$ is $du = 3x^2dx$, (if $u = f(x)$ then $du = f'(x)dx$).

$$\frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2dx$$

Now we rewrite the integral:

$$\int 3x^2\sqrt{1+x^3}dx = \int \sqrt{\underbrace{1+x^3}_{u}}\ \underbrace{3x^2dx}_{du} = \int \sqrt{u}\,du$$

$$= \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(1+x^3)^{3/2} + C$$

Is this correct? Let's check using the chain rule:

$$\frac{d}{dx}\left[\frac{2}{3}(1+x^3)^{3/2}+C\right] = \frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(1+x^3)^{1/2}\cdot(3x^2) + 0$$

$$= 3x^2\left(1+x^3\right)^{1/2}$$

But why does this work?
Remember that the chain rule says:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

But this means that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

$$\int f'(g(x))g'(x)dx = \int f'(u)\,du = f(u)+C = f(g(x))+C$$

LET $u = g(x) \Rightarrow du = g'(x)dx$

The substitution rule is undoing the chain rule.

**<u>The Substitution Rule</u>** Suppose $g(x)$ is a differentiable function and $f(x)$ is a continuous function. Then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $u = g(x)$.

Example: Find the following indefinite intedgrals:

1) $\int 2x(x^2+3)^4 dx = \int (\underbrace{x^2+3}_{u})^4 \underbrace{2x\,dx}_{du}$

$= \int u^4 du = \frac{u^5}{5} + C = \frac{(x^2+3)^5}{5} + C$

LET
$u = x^2+3$
$du = 2x\,dx$

2) $\int x^2(x^3+1)^{3/2}dx = \int (x^3+1)^{3/2}\, x^2 dx$

LET
$u = x^3+1$
$du = 3x^2 dx$

$= \int \frac{(x^3+1)^{3/2}\, 3x^2 dx}{3} = \frac{1}{3}\int \underbrace{(x^3+1)}_{u}{}^{3/2}\, \underbrace{3x^2 dx}_{du}$

$= \frac{1}{3}\int u^{3/2} du = \frac{1}{3}\frac{u^{5/2}}{5/2} + C$

$= \frac{2}{15}u^{5/2} + C = \frac{2}{15}(x^3+1)^{5/2} + C$

Note: When finding *du* we can isolate for *dx* and substitute into the integral. Let's look at the last example again:

$\int x^2(x^3+1)^{3/2} dx = \int \cancel{x^2}(u)^{3/2}\frac{du}{3\cancel{x^2}}$

LET
$u = x^3+1$
$du = 3x^2 dx$ ISOLATE $dx$
$\frac{du}{3x^2} = dx$

$= \frac{1}{3}\int u^{3/2} du = \frac{1}{3}\frac{u^{5/2}}{5/2} + C$

$= \frac{2}{15}(x^3+1)^{5/2} + C$

NOTE: WHEN YOU ARE FINISHED WITH YOUR SUBSTITUTION THERE SHOULD ONLY BE ONE VARIABLE!

3) $\int \frac{x}{1+3x^2}dx = \int \frac{\cancel{x}}{u} \frac{du}{6\cancel{x}}$

$= \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + c$

$= \frac{1}{6}\ln|1+3x^2| + C$

LET
$u = 1+3x^2$
$du = 6x\,dx$
$\frac{du}{6x} = dx$

4) $\int x^4 \sin(x^5+3)dx = \int \cancel{x^4} \sin u \frac{du}{5\cancel{x^4}}$

$= \frac{1}{5}\int \sin u\,du = -\frac{1}{5}\cos u + C$

$= -\frac{1}{5}\cos(x^5+3) + C$

LET
$u = x^5+3$
$du = 5x^4dx$
$\frac{du}{5x^4} = dx$

5) $\int \frac{(\ln x)^3}{4x}dx = \int \frac{u^3}{4\cancel{x}}\cancel{x}\,du$

$= \frac{1}{4}\int u^3 du = \frac{1}{4}\frac{u^4}{4} + C$

$= \frac{1}{16}(\ln x)^4 + C$

LET
$u = \ln x$
$du = \frac{1}{x}dx$
$x\,du = dx$

6) $\int \frac{e^{-1/x}}{x^2}dx$ $= \int \frac{e^u \cancel{x^2}\,du}{\cancel{x^2}}$

$= \int e^u\,du = e^u + C = e^{-1/x} + C$

$= \int \frac{u}{\cancel{x^2}} \frac{\cancel{x^2}\,du}{e^{-1/x}} = \int \frac{u}{e^{-1/x}}\,du\ ?$

$= \int \frac{u}{u}\,du = \int 1\,du = u + C = e^{-1/x} + C$

LET
$u = -1/x = -x^{-1}$
$du = x^{-2}\,dx$
$\frac{du}{x^{-2}} = dx$
$x^2\,du = dx$

LET $u = e^{-1/x}$
$du = e^{-1/x} x^{-2}\,dx$
$\frac{du\,x^2}{e^{-1/x}} = dx$

7) $\int \frac{3x^2+2}{(x^3+2x)^2}dx$ $= \int \frac{\cancel{3x^2+2}}{u^2} \cdot \frac{du}{\cancel{3x^2+2}}$

$= \int u^{-2}\,du = -u^{-1} + C$

$= -(x^3+2x)^{-1} + C$

LET
$u = x^3 + 2x$
$du = (3x^2+2)\,dx$
$\frac{du}{3x^2+2} = dx$

Note: We may need to rearange the integrand in order to make the substitution work out.

8) $\int x\sqrt{x+5}\,dx$ $= \int x\sqrt{u}\,du\ ?$

$= \int (u-5)\underbrace{\sqrt{u}}_{u^{1/2}}\,du = \int (u^{3/2} - 5u^{1/2})\,du$

$= \frac{u^{5/2}}{5/2} - \frac{5u^{3/2}}{3/2} + C = \frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2} + C$

LET
$u = x + 5$
$du = dx$
$u - 5 = x$

9) $\int \frac{x+2}{\sqrt{x}-1}dx = \int \frac{x+2}{u} \cdot 2\sqrt{x}\, du$

Let $u = \sqrt{x} - 1$

$du = \frac{1}{2}x^{-1/2}dx$

$du = \frac{dx}{2\sqrt{x}}$

$2\sqrt{x}\,du = dx$

$u+1 = \sqrt{x}$

$(u+1)^2 = x$

$= \int \frac{[(u+1)^2+2]\cdot 2(u+1)}{u} du = \int \frac{(u^2+2u+1+2)\cdot(2u+2)}{u} du$

$= \int \frac{(u^2+2u+3)(2u+2)}{u} du = \int \frac{2u^3+4u^2+6u+2u^2+4u+6}{u} du$

$= \int \frac{2u^3+6u^2+10u+6}{u} du = \int 2u^2+6u+10+6u^{-1}\, du$

$= \frac{2u^3}{3} + 3u^2 + 10u + 6\ln|u| + C = \frac{2}{3}(\sqrt{x}-1)^3 + 3(\sqrt{x}-1)^2 + 10(\sqrt{x}-1) + 6\ln|\sqrt{x}-1| + C$

Let's examine where some of the trig integrals that we saw last class came from.

$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{\cancel{\sin x}}{u} \frac{du}{-\cancel{\sin x}}$

Let $u = \cos x$

$du = -\sin x\, dx$

$\frac{du}{-\sin x} = dx$

$= -\int \frac{1}{u} du = -\ln|u| + C$

$= -\ln|\cos x| + C$

$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)dx$

$\frac{du}{\sec x \tan x + \sec^2 x} = dx$

$= \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x} dx =$

$= \int \frac{\sec^2 x \cancel{+ \tan x} \sec x}{u} \frac{du}{\sec x \cancel{\tan x} + \sec^2 x}$

$= \int \frac{1}{u} du = \ln|u| + C = \ln|\sec x + \tan x| + C$

## Substituting for linear functions

If we're making the substitution $u = ax + b$ we can often do most of it in our heads. Since $du = a\,dx$ which means $\frac{1}{a}du = dx$.
Example:

1) $\int \cos 3x\,dx = \int \cos u \frac{du}{3}$

$= \frac{1}{3}\int \cos u\, du = \frac{1}{3}\sin u + C$

$= \frac{1}{3}\sin 3x + C$

LET $u = 3x$

$du = 3dx$

$\frac{du}{3} = dx$

2) $\int e^{7x+1}\,dx = \frac{1}{7}e^{7x+1} + C$

$= \int e^u \frac{du}{7} = \frac{1}{7}\int e^u du = \frac{1}{7}e^u + C$

$= \frac{1}{7}e^{7x+1} + C$

LET

$u = 7x + 1$

$du = 7dx$

$\frac{du}{7} = dx$

3) $\int \sec^2 6x\,dx = \frac{1}{6}\tan 6x + C$

4) $\int \sec(-2x)\tan(-2x)\,dx = \frac{1}{-2}\sec(-2x) + C$

$\int \sec u \tan u \frac{du}{-2} =$

$-\frac{1}{2}\int \sec u \tan u\, du = -\frac{1}{2}\sec u + C$

$= -\frac{1}{2}\sec(-2x) + C$

LET $u = -2x$

$du = -2dx$

$\frac{du}{-2} = dx$