Dawson College: **Linear Algebra**: 201-105-dw-S03: Fall 2012

Name: Y. Lamontagne
Student ID: ____________

Test 1

This test is graded out of 55 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(10 marks)* Solve the following system by Gauss-Jordan elimination:

$$
\begin{array}{rcrcrcrcrcl}
3x_1 & - & x_2 & + & x_3 & - & 3x_4 & + & x_5 & = & 3 \\
4x_1 & + & 3x_2 & - & x_3 & + & x_4 & - & x_5 & = & 2 \\
7x_1 & + & 2x_2 & & & - & 2x_4 & - & \frac{1}{3}x_5 & = & 1
\end{array}
$$

$$
\begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 4 & 3 & -1 & 1 & -1 & 2 \\ 7 & 2 & 0 & -2 & -\frac{1}{3} & 1 \end{bmatrix}
$$

$$
\sim \begin{array}{r} \\ 3R_2 \\ 3R_3 \end{array}
\begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 12 & 9 & -3 & 3 & -3 & 6 \\ 21 & 6 & 0 & -6 & -1 & 3 \end{bmatrix}
$$

$$
\sim \begin{array}{r} \\ -4R_1+R_2\to R_2 \\ -7R_1+R_3\to R_3 \end{array}
\begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 0 & 13 & -7 & 15 & -7 & -6 \\ 0 & 13 & -7 & 15 & -8 & -18 \end{bmatrix}
$$

$$
\sim \begin{array}{r} \\ \\ -R_2+R_3\to R_3 \end{array}
\begin{bmatrix} 3 & -1 & 1 & -3 & 1 & 3 \\ 0 & 13 & -7 & 15 & -7 & -6 \\ 0 & 0 & 0 & 0 & -1 & -12 \end{bmatrix}
$$

$$
\sim \begin{array}{r} R_3+R_1\to R_1 \\ -7R_3+R_2\to R_2 \\ -R_3\to R_3 \end{array}
\begin{bmatrix} 3 & -1 & 1 & -3 & 0 & -9 \\ 0 & 13 & -7 & 15 & 0 & 78 \\ 0 & 0 & 0 & 0 & 1 & 12 \end{bmatrix}
$$

$$
\sim \begin{array}{r} 13R_1 \\ \\ \\ \end{array}
\begin{bmatrix} 39 & -13 & 13 & -39 & 0 & -117 \\ 0 & 13 & -7 & 15 & 0 & 78 \\ 0 & 0 & 0 & 0 & 1 & 12 \end{bmatrix}
$$

$$
\sim \begin{array}{r} R_2+R_1\to R_1 \\ \\ \\ \end{array}
\begin{bmatrix} 39 & 0 & 6 & -24 & 0 & -39 \\ 0 & 13 & -7 & 15 & 0 & 78 \\ 0 & 0 & 0 & 0 & 1 & 12 \end{bmatrix}
$$

$$
\sim \begin{array}{r} \frac{1}{39}R_1 \\ \frac{1}{13}R_2 \\ \\ \end{array}
\begin{bmatrix} 1 & 0 & 2/13 & -8/13 & 0 & -1 \\ 0 & 1 & -7/13 & 15/13 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 12 \end{bmatrix}
$$

Let

$x_3 = s$
$x_4 = t$

$\therefore\ x_1 = -\frac{2}{13}s + \frac{8}{13}t - 1$

$x_2 = \frac{7}{13}s - \frac{15}{13}t + 6$

$x_5 = -12$

Question 2. Consider the matrices:

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 4 & -3 \\ 2 & -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & -4 & 0 \\ 2 & 2 & 1 \end{bmatrix} C = \begin{bmatrix} 0 & 1 \\ -3 & 2 \\ 0 & -1 \end{bmatrix} D = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} E = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

a. *(2 marks)* Compute the following, if possible.

E^{-1} not invertible since one of the diagonal entry is 0.

b. *(2 marks)* Compute the following, if possible.

$D - BC$

$$D-BC = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -4 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 12 & -8 \\ -6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 7 \\ 5 & -5 \end{bmatrix}$$

c. *(2 marks)* Compute the following, if possible.

$tr(E) = 2+0+3 = 5$

d. *(3 marks)* Compute the following, if possible.

$C^tB^t = (BC)^t = \begin{bmatrix} 12 & -6 \\ -8 & 5 \end{bmatrix}$

e. *(3 marks)* Compute the following, if possible.

AB^t

$$AB^t = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 4 & -3 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -4 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -16 & 5 \\ 4 & 0 \end{bmatrix}$$

(5 marks) Find F, if possible.

$(F^{-1}+2I)^{-1} = (BC)^t$

$$(F^{-1}+2I)^{-1} = \begin{bmatrix} 12 & -6 \\ -8 & 5 \end{bmatrix}$$

$$\left((F^{-1}+2I)^{-1}\right)^{-1} = \begin{bmatrix} 12 & -6 \\ -8 & 5 \end{bmatrix}^{-1}$$

$$F^{-1} + 2I = \frac{1}{12}\begin{bmatrix} 5 & 6 \\ 8 & 12 \end{bmatrix}$$

$$F^{-1} = \begin{bmatrix} 5/12 & \frac{1}{2} \\ 2/3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F^{-1} = \begin{bmatrix} -19/12 & 1/2 \\ 2/3 & -1 \end{bmatrix}$$

$$F = \begin{bmatrix} -19/12 & 1/2 \\ 2/3 & -1 \end{bmatrix}^{-1}$$

$$F = \frac{1}{\frac{19}{12} - \frac{2}{6}}\begin{bmatrix} -1 & -1/2 \\ -2/3 & -19/12 \end{bmatrix}$$

$$\frac{12}{15}\begin{bmatrix} -1 & -1/2 \\ -2/3 & -19/12 \end{bmatrix}$$

$$= \begin{bmatrix} -12/15 & -6/15 \\ -8/15 & -19/15 \end{bmatrix}$$

Question 3. *(5 marks)* Solve the following system by Gaussian elimination:

$$\begin{aligned} 2x_1 - x_2 - x_3 - 3x_4 &= 0 \\ 3x_1 + 2x_2 - x_3 + x_4 &= 1 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & -1 & -3 & 0 \\ 3 & 2 & -1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ 2R_2 \end{array} \begin{bmatrix} 2 & -1 & -1 & -3 & 0 \\ 6 & 4 & -2 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ -3R_1+R_2 \to R_2 \end{array} \begin{bmatrix} 2 & -1 & -1 & -3 & 0 \\ 0 & 7 & 1 & 11 & 2 \end{bmatrix}$$

$$\sim \begin{array}{c} \frac{1}{2}R_1 \\ \frac{1}{7}R_2 \end{array} \begin{bmatrix} 1 & -1/2 & -1/2 & -3/2 & 0 \\ 0 & 1 & 1/7 & 11/7 & 2/7 \end{bmatrix}$$

Let $x_3 = s$

$x_4 = t$

and sub into below

$x_1 - \frac{1}{2}x_2 - \frac{1}{2}s - \frac{3}{2}t = 0$ (2)

$x_2 + \frac{1}{7}s + \frac{11}{7}t = 2/7$

$x_2 = \frac{2}{7} - \frac{1}{7}s - \frac{11}{7}t$ (1)

sub (1) into (2)

$x_1 = \frac{1}{2}\left(\frac{2}{7} - \frac{1}{7}s - \frac{11}{7}t\right) + \frac{1}{2}s + \frac{3}{2}t$

$x_1 = \frac{1}{7} + \frac{6}{14}s + \frac{10}{14}t$

$\therefore$ $x_1 = \frac{1}{7} + \frac{3}{7}s + \frac{5}{7}t$

$x_2 = \frac{2}{7} - \frac{1}{7}s - \frac{11}{7}t$

$x_3 = s$

$x_4 = t$

Question 4. *(6 marks)* Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and A^{-1} as a product of elementary matrices. *Explicitely write all elementary matrices.*

$$A \sim_{\underset{-2R_1+R_2\to R_2}{①}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim_{\underset{R_3+R_2\to R_2}{②}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim_{\underset{-\frac{1}{2}R_2\to R_2}{③}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

①: $I \sim_{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1 \quad \therefore E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

②: $I \sim_{R_3+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = E_2 \quad \therefore E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

③: $I \sim_{-\frac{1}{2}R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3 \quad \therefore E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $\quad E_3E_2E_1A = I$

$\therefore A^{-1} = E_3E_2E_1$

$A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$

Question 5. *(7 marks)* Solve the system by using and inverting the coefficient matrix.

$$\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 4 \\ 2x & - & 5y & + & z & = & -1 \\ 2x & - & 6y & + & 3z & = & 3 \end{array}$$

$Ax = b$

where $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -5 & 1 \\ 2 & -6 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -3 & 1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 & 1 & 0 \\ 2 & -6 & 3 & 0 & 0 & 1 \end{array}\right] \sim \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -R_3 + R_1 \to R_1 \\ R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \sim \; 3R_2 + R_1 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & 3 & 2 \\ 0 & 1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

So we have $Ax = b$

$x = A^{-1}b$

$$x = \begin{bmatrix} -9 & 3 & 2 \\ -4 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -33 \\ -14 \\ -5 \end{bmatrix}$$

Question 6. A square matrix A is said to be *skew-symmetric* if $A^T = -A$. Prove:

a. *(2 marks)* If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b. *(3 marks)* If A and B are skew-symmetric matrices, then so are A^T, $A \pm B$, and kA for any scalar.

a) Premise: ① $AA^{-1} = I = A^{-1}A$
② $A^T = -A$

WTS: $(A^{-1})^T = -A$

$$\begin{aligned} LHS &= (A^{-1})^T \\ &= (A^T)^{-1} \\ &= (-A)^{-1} \\ &= -A^{-1} \\ &= RHS \end{aligned}$$

$\blacksquare$

b) Premise: $A^T = -A \quad \therefore A = -A^T$
$B^T = -B$

Want to show: ① $(A^T)^T = -A^T$
② $(A \pm B)^T = -(A \pm B)$
③ $(KA)^T = -KA$

① $LHS = (A^T)^T = A = -A^T = RHS$ $\blacksquare$
② $LHS = (A \pm B)^T = A^T \pm B^T = -A \pm -B = -(A \pm B) = RHS$ $\blacksquare$
③ $LHS = (KA)^T = KA^T = K(-A) = -KA = RHS$ $\blacksquare$

Question 7. *(5 marks)* Determine the conditions on the b_i's, if any, in order to guarantee that the linear system is consistent

$$\begin{aligned} 3x_1 - 2x_2 + 3x_3 &= 1 \\ 6x_1 - 6x_2 + 6x_3 &= b_1 \\ -3x_1 + 2x_2 - 3x_3 &= b_2 \end{aligned}$$

$$\begin{bmatrix} 3 & -2 & 3 & 1 \\ 6 & -6 & 6 & b_1 \\ -3 & 2 & -3 & b_2 \end{bmatrix} \sim \begin{matrix} -2R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{matrix} \begin{bmatrix} 3 & -2 & 3 & 1 \\ 0 & -2 & 0 & b_1 - 2 \\ 0 & 0 & 0 & b_2 + 1 \end{bmatrix}$$

In order for the system to be consistent there needs to be no contradiction

i.e.

$b_2 + 1 = 0$

$b_2 = -1$

Bonus Question. *(5 marks)* Prove that if $ad - bc \neq 0$, then the reduced row echelon form of

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

If $ad - bc \neq 0$ then $a \neq 0$ or $c \neq 0$

Suppose $a \neq 0$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim_{aR_2} \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \sim_{-cR_1 + R_2 \to R_2} \begin{bmatrix} a & b \\ 0 & ad-bc \end{bmatrix}$$

$$\sim_{\frac{1}{ad-bc}R_2} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$\sim_{-bR_2 + R_1 \to R_1} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sim_{\frac{1}{a}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose $c \neq 0$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ a & b \end{bmatrix} \sim_{cR_2} \begin{bmatrix} c & d \\ ac & cb \end{bmatrix}$$

$$\sim_{-aR_1 + R_2 \to R_2} \begin{bmatrix} c & d \\ 0 & cb-ad \end{bmatrix}$$

$$\sim_{\frac{1}{cb-ad}R_2} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}$$

$$\sim_{-dR_2 + R_1 \to R_1} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sim_{\frac{1}{c}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$