Dawson College: **Linear Algebra**: 201-105-dw-S03: Fall 2012

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Test 2

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

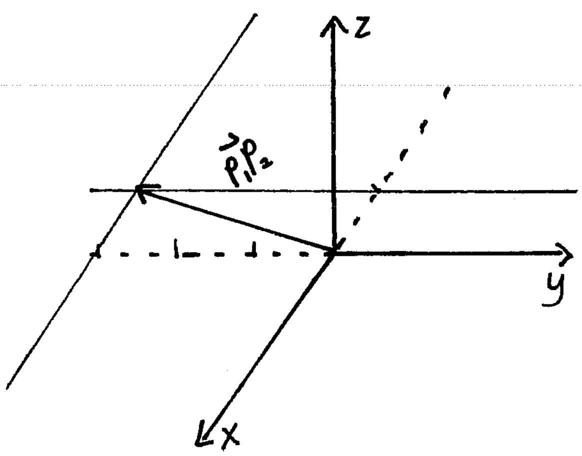
$$P_1(-1,2,1),\ P_2(-2,-1,1),\ \mathbf{u}=(-1,2,-3),\ \mathbf{v}=(3,1,4),\ \mathbf{w}=(5,-1,-2)$$

a. *(2 marks)* Sketch the vector $\overrightarrow{P_1P_2}$ with the initial point located at the origin.

b. *(4 marks)* Find the angle θ in radians between $\mathbf{u}$ and $\mathbf{v}$.

c. *(4 marks)* Find two unit vectors orthogonal to $\mathbf{w}$.

a) $\vec{P_1P_2} = P_2 - P_1 = (-2,-1,1)-(-1,2,1) = (-1,-3,0)$



b) $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$\cos\theta = \dfrac{(-1,2,-3)\cdot(3,1,4)}{\|(-1,2,3)\|\|(3,1,4)\|}$

$\cos\theta = \dfrac{(-1)(3)+2(1)+(-3)(4)}{\sqrt{3^2+1^2+4^2}\sqrt{(-1)^2+2^2+3^2}}$

$\cos\theta = \dfrac{-3+2-12}{\sqrt{9+1+16}\sqrt{1+4+9}}$

$\cos\theta = \dfrac{-13}{\sqrt{14}\sqrt{26}}$

$\theta = \arccos\left(\dfrac{-13}{\sqrt{14}\sqrt{26}}\right)$

$\theta \doteq 2.32$

c) Let $\vec{c} = (x_1, y_1, z_1)$

$\vec{w}\cdot\vec{c} = 0$

$0 = (5,-1,-2)\cdot(x_1,y_1,z_1)$

$0 = 5x_1 - y_1 - 2z_1$

Let $x_1 = y_1 = 1$

$0 = 5(1) - 1 - 2z_1$

$2z_1 = 4$

$z_1 = 2$

$\therefore \vec{c} = (1,1,2) \perp \vec{w}$

unit vector $\parallel \vec{c} = \dfrac{1}{\|\vec{c}\|}\vec{c}$

$= \dfrac{1}{\sqrt{1^2+1^2+2^2}}(1,1,2)$

$= \dfrac{1}{\sqrt{6}}(1,1,2) = \left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{2}{\sqrt6}\right)$

& an other unit vector $\perp \vec{w}$

$= -\left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{2}{\sqrt6}\right)$

1c) unit vector with same diretion as $\vec{w}$ $= \frac{1}{\|\vec{w}\|}\vec{w} = \frac{1}{\sqrt{5^2+(-1)^2+(-2)^2}}(5,-1,-2) = \frac{1}{\sqrt{30}}(5,-1,-2)$

$= \left(\frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}\right)$

unit vector with opposite direction as $\vec{u}$ $= \frac{1}{\|\vec{u}\|}(-\vec{u}) = \frac{-1}{\sqrt{(-1)^2+2^2+(-3)^2}}(-1,2,-3) = \frac{-1}{\sqrt{14}}(-1,2,-3)$

$= \left(\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Question 2. *(2 marks)* Show that if A is a square matrix, then

$\det(A^TA) = \det(AA^T)$

$LHS = det(A^TA) = det(A^T)det(A) = det(A)det A^T = det(AA^T) = RHS$

Question 3. *(2 marks)* Prove or disprove*(using an example)*: If A and B are square matrices then

$\det(A+B) = \det(A) + \det(B)$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$det(A+B)$

$det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \stackrel{?}{=} det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$1 \stackrel{?}{=} 1 - 1$

$1 = 0$

Question 3. *(5 marks)* Solve using Cramer's rule.

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ x_1 + 3x_2 - 5x_3 &= 1 \\ -2x_1 - x_2 - x_3 &= -2 \end{aligned}$$

Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & -5 \\ -2 & -1 & -1 \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= 3(-1)^{1+1} \begin{vmatrix} 3 & -5 \\ -1 & -1 \end{vmatrix}$

$+ (-1)(-1)^{1+2} \begin{vmatrix} 1 & -5 \\ -2 & -1 \end{vmatrix}$

$+ (1)(-1)^{1+3} \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}$

$= 3[3(-1) - (-5)(-1)]$

$+ [(1)(-1) - (-5)(-2)]$

$+ [1(-1) - 3(-2)]$

$= 3[-8] + [-11] + [5]$

$= -30$

$A_1 = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & -5 \\ -2 & -1 & -1 \end{bmatrix}$, $|A_1| = -30$

$A_2 = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 1 & -5 \\ -2 & -2 & -1 \end{bmatrix}$, $|A_2| = 0$

$A_3 = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 3 & 1 \\ -2 & -1 & -2 \end{bmatrix}$, $|A_3| = 0$

$\therefore X_1 = \frac{|A_1|}{|A|} = \frac{-30}{-30} = 1$, $X_2 = \frac{|A_2|}{|A|} = \frac{0}{-30} = 0$

$X_3 = \frac{|A_3|}{|A|} = \frac{0}{-30} = 0$

Question 4. *(4 marks)* Let A and C be invertible matrices of the same size such that $\det(A) = 4$. Find $\det(C)$ if, $A^2 = C^2A^T$.

$$A^2 = C^2A^T$$
$$\det(A^2) = \det(C^2A^T)$$
$$[\det A]^2 = \det[C^2]\det[A^T]$$
$$4^2 = (\det(C))^2 \det A$$
$$4^2 = (\det(C))^2\, 4$$
$$4 = (\det C)^2$$
$$\pm\sqrt{4} = \det C$$
$$\pm 2 = \det C$$

Question 5. *(4 marks)* Let A and B be 5×5 matrices such that $\det(A) = 2$, $\det(B) = 3$. Find $\det\left(\det(A)B^2 - 3B^2\right)$.

$$\det(\det(A)B^2 - 3B^2)$$
$$= \det((\det(A)I - 3I)B^2)$$
$$= \det((\det(A) - 3)I\, B^2)$$
$$= \det((\det(A) - 3)I)\det B^2$$
$$= (\det(A) - 3)^5 \det(I)(\det B)^2$$
$$= (2-3)^5 (1)(3)^2$$
$$= (-1)^5(1)(3)^2$$
$$= -9$$

or

$$\det(2B^2 - 3B^2)$$
$$= \det(-B^2)$$
$$= (-1)^5 \det(B^2)$$
$$= -(\det B)^2 = -(3)^2 = -9$$

Question 6. *(5 marks)* Evaluate the determinant

$$\begin{vmatrix} 5d & -a & 4g-7a \\ 5e & -b & 4h-7b \\ 5f & -c & 4i-7c \end{vmatrix}$$

given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$$

$$= \begin{vmatrix} 5d & 5e & 5f \\ -a & -b & -c \\ 4g-7a & 4h-7b & 4i-7c \end{vmatrix}$$

$$= 5(-1) \begin{vmatrix} d & e & f \\ a & b & c \\ 4g-7a & 4h-7b & 4i-7c \end{vmatrix}$$

$$= 5(-1)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix} \qquad \begin{matrix} R_1 \leftrightarrow R_2 \\ 7R_2 + R_3 \rightarrow R_3 \end{matrix}$$

$$= 5(4) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= 20(2)$$

$$= 40$$

Question 7. Given

$$A = \begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & 10 & -8 \\ -7 & 1 & -2 \\ 3 & -5 & 4 \end{bmatrix}$$

a. *(5 marks)* Is A invertible, justify.

b. *(2 marks)* Is B invertible, justify.

$$A \sim \begin{array}{r} \\ 2R_1 + R_2 \to R_2 \\ \\ -2R_3 + R_4 \to R_4 \\ \\ \end{array} \begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ \\ \\ -R_4 + R_5 \to R_5 \end{array} \begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = C$$

det [new matrix] = (operation that change the det.) det [original matrix]

$$\det C = (1)(1)(1) \det A$$
$$(1)(-1)(1)(1)(2) = \det A$$
$$-2 = \det A$$

$\therefore A$ is invertible.

b) $\det B = 0$ since $R_1 = -2R_3$

$\therefore B$ is not invertible.

Question 8. *(5 marks)* Solve for x

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

$$x(1-x) - (-1)(3) = a_{11}C_{11} + \overset{0}{a_{12}C_{12}} + a_{13}C_{13}$$

$$x(1-x) + 3 = 1(-1)^{1+1}\begin{vmatrix} x & -6 \\ 3 & x-5 \end{vmatrix} + (-3)(-1)^{1+3}\begin{vmatrix} 2 & x \\ 1 & 3 \end{vmatrix}$$

$$x - x^2 + 3 = [x(x-5) - (-6)(3)] - 3[2(3) - x]$$

$$x - x^2 + 3 = x^2 - 5x \cancel{+18} \cancel{-18} + 3x$$

$$0 = 2x^2 - 3x - 3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+24}}{4} = \frac{3 \pm \sqrt{33}}{4}$$

Question 9. *(6 marks)* Decide whether the given matrix is invertible, and if so, use the adjoint method to find its inverse.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -2 & 5 & 6 \end{bmatrix}$$

$|A| = 2(-1)(6) = -12$

$\therefore A$ is invertible

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 5 & 6 \end{vmatrix} = -6$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 0 \\ -2 & 6 \end{vmatrix} = -18$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix} = 13$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ 5 & 6 \end{vmatrix} = 0$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 0 \\ -2 & 6 \end{vmatrix} = 12$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 0 \\ -2 & 5 \end{vmatrix} = -10$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} = -2$$

$$\text{matrix of cofactor} = \begin{bmatrix} -6 & -18 & 13 \\ 0 & 12 & -10 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -6 & 0 & 0 \\ -18 & 12 & 0 \\ 13 & -10 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-12}\begin{bmatrix} -6 & 0 & 0 \\ -18 & 12 & 0 \\ 13 & -10 & -2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 3/2 & -1 & 0 \\ -13/12 & 5/6 & 1/6 \end{bmatrix}$$

Bonus Question. *(5 marks)* Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^2 = AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$$tr(A) = a+d$$

$$tr(A^2) = a^2+bc+bc+d^2$$

$$= a^2+2bc+d^2$$

$$|A| = ad-bc$$

$$RHS = \frac{1}{2}\begin{vmatrix} trA & 1 \\ trA^2 & trA \end{vmatrix} = \frac{1}{2}\left[(trA)^2 - trA^2\right]$$

$$= \frac{1}{2}\left[(a+d)^2 - (a^2+2bc+d^2)\right]$$

$$= \frac{1}{2}\left[a^2+2ad+d^2-a^2-2bc-d^2\right]$$

$$= \frac{1}{2}\left[2ad-2bc\right] = ad-bc = |A|$$

$\blacksquare$