**Dawson College**: **Linear Algebra**: 201-105-dw-S03: Fall 2012

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# Test 3

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

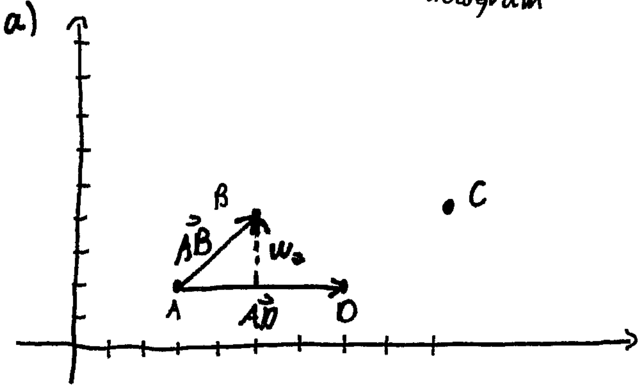
**Question 1.** Given the following vertices $A(3,2)$, $B(5,4)$, $C(9,4)$, $D(7,2)$.

a. *(3 marks)* Find the area of the parallelogram $ABCD$ using the determinant.

b. *(3 marks)* Find the length of the altitude from the side $AD$ to side $BC$ using projections.

c. *(1 mark)* Find the area of the parallelogram $ABCD$ using part b.

a)




$\therefore$ the parallelogram is determined by $\vec{u}=\vec{AB}$ and $\vec{v}=\vec{AD}$

$\vec{u}=\vec{AB}=B-A=(5,4)-(3,2)=(2,2)$
$\vec{v}=\vec{AD}=D-A=(7,2)-(3,2)=(4,0)$

$$\text{Area} = \left\|\begin{matrix} u_1 & u_2 \\ v_1 & v_2 \end{matrix}\right\|$$

$$= \left\|\begin{matrix} 2 & 2 \\ 4 & 0 \end{matrix}\right\| = |-8| = 8$$

b) length of the altitude from AD to BC $= \|\vec{w}_2\|$

$$\vec{w}_2 = \vec{AB} - \text{proj}_{\vec{AD}}\vec{AB}$$
$$= (2,2) - \frac{(2,2)\cdot(4,0)}{(4,0)\cdot(4,0)}(4,0)$$
$$= (2,2) - \frac{8}{16}(4,0)$$
$$= (0,2)$$

$\therefore$ length $= \|\vec{w}_2\| = \|(0,2)\| = 2$

c) Area = base $\cdot$ height
$= \|\vec{AD}\|\|\vec{w}_2\|$
$= \|(4,0)\|\|(0,2)\|$
$= 4\cdot 2$
$= 8$

**Question 2.** *(2 marks)* Given

$$\begin{array}{lrcll}
\mathscr{L}_1: & (x,y,z) & = & (2+5t,1+t,-t) & t\in\mathbb{R}\\
\mathscr{L}_2: & (x,y,z) & = & (7+2t,4,10t) & t\in\mathbb{R}\\
\mathscr{L}_3: & (x,y,z) & = & (9-t,2,9-5t) & t\in\mathbb{R}\\
\mathscr{P}_1: & x-2y+3z-11 & = & 0 & \\
\mathscr{P}_2: & -5x-y+z+31 & = & 0 & \\
\mathscr{P}_3: & -3x+6y-9z+1 & = & 0 &
\end{array}$$

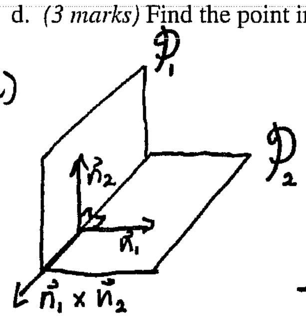
a. *(2 marks)* Find a vector parallel to the intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$.

b. *(2 marks)* Find a unit vector perpendicular to $\mathscr{P}_3$.

c. *(1 marks)* Find a vector parallel to $\mathscr{L}_3$.

d. *(3 marks)* Find the point intersection between $\mathscr{P}_2$ and $\mathscr{L}_1$ if it exists.

a)

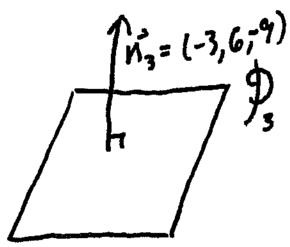



where $\vec{n}_1 = (1,-2,3)$
$\vec{n}_2 = (-5,-1,1)$

$$\vec{n}_1 \times \vec{n}_2 = \left( \begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -5 \\ -2 & -1 \end{vmatrix} \right)$$

$$\begin{array}{cc} 1 & -5 \\ -2 & -1 \\ 3 & 1 \end{array} = (1,-16,-11)$$

b)

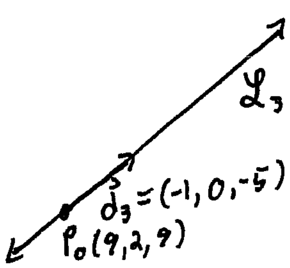



So $\vec{u} = \dfrac{\vec{n}_3}{\|\vec{n}_3\|} = \dfrac{(-3,6,-9)}{\sqrt{(-3)^2+6^2+(-9)^2}} = \dfrac{(-3,6,-9)}{\sqrt{126}}$

$$= \frac{(-3,6,-9)}{3\sqrt{14}}$$

$$= \left( \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right)$$

c)




$\therefore$ vector parallel to line is $(-1,0,-5)$.

d) $\mathscr{L}_1 \begin{cases} x = 2+5t \\ y = 1+t \\ z = -t \end{cases}$

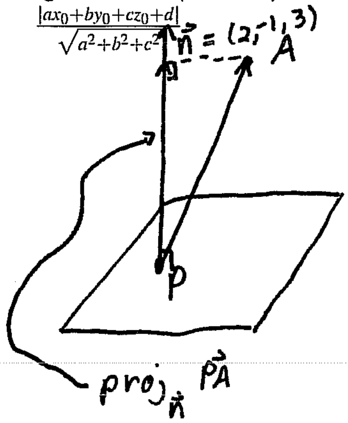
sub into $\mathscr{P}_2$

$$-5(2+5t) - (1+t) + (-t) + 31 = 0$$
$$-27t = -20$$
$$t = \frac{20}{27}$$

$\therefore$ intersection point is

$$x = 2+5\left(\frac{20}{27}\right) = \frac{154}{27}$$
$$y = 1 + \frac{20}{27} = \frac{47}{27}$$
$$z = -\frac{20}{27}$$

**Question 3.** *(4 marks)* Find the distance between $A(1,0,3)$ and the plane $2x - y + 3z + 3 = 0$. **Do NOT use the formula:** $D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$




$\therefore$ $d = \| proj_{\vec{n}} \vec{PA} \|$

Lets find a point on the plane.

Let $x = z = 0$ : $2(0) - y + 3(0) + 3 = 0$

$-y = -3$

$y = 3$ $\therefore P(0,3,0)$

and $\vec{PA} = A - P = (1,0,3) - (0,3,0) = (1,-3,3)$

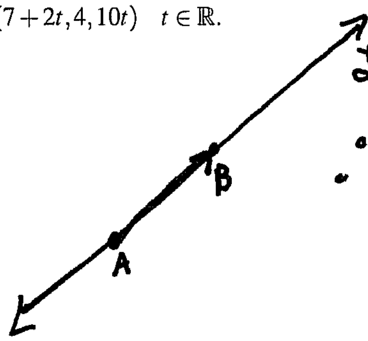
$d = \| proj_{\vec{n}} \vec{PA} \|$

$= \left\| \frac{(1,-3,3)\cdot(2,-1,3)}{(2,-1,3)\cdot(2,-1,3)} (2,-1,3) \right\|$

$= \left\| \frac{14}{14}(2,-1,3) \right\| = \frac{14}{14}\sqrt{2^2+(-1)^2+3^2}$

$= \sqrt{14}$

**Question 4.** *(6 marks)* Determine whe$\overset{h}{\vee}$ter the line $\mathscr{L}_1$ passing through the point $A(2,3,3)$ and $B(3,1,8)$ intersects $\mathscr{L}_2 : \quad (x,y,z) = (7+2t, 4, 10t) \quad t \in \mathbb{R}$.




$\mathscr{L}_1 : (x,y,z) = A + s\vec{AB} \quad s \in \mathbb{R}$ where

$\vec{AB} = B - A$

$= (3,1,8) - (2,3,3)$

$= (1,-2,5)$

$\therefore \mathscr{L}_1 \begin{cases} x = 2 + s \\ y = 3 - 2s \\ z = 3 + 5s \end{cases}$

Let $\mathscr{L}_1 = \mathscr{L}_2$

① $2 + s = 7 + 2t$

② $3 - 2s = 4$

③ $3 + 5s = 10t$

From ② $s = -\frac{1}{2}$ sub in ①

$2 - \frac{1}{2} = 7 + 2t$

$2 - \frac{1}{2} - 7 = 2t$

$t = -\frac{11}{4}$

Lets verify consistency by subbing in ③

$3 + 5(-\frac{1}{2}) = 10(-\frac{11}{4})$

$\frac{1}{2} = -\frac{110}{4}$

$\frac{1}{2} = -\frac{55}{2}$ ✗

$\therefore$ no intersection

**Question 5.** *(5 marks)* Given

$$\begin{aligned} \mathscr{P}_1: \quad -x-3y+2z-3 &= 0 \\ \mathscr{P}_2: \quad 2x+4y-3z+5 &= 0, \end{aligned}$$

find the intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$.

$$\begin{aligned} -x-3y+2z &= 3 \\ 2x+4y-3z &= -5 \end{aligned}$$

$$\begin{bmatrix} -1 & -3 & 2 & 3 \\ 2 & 4 & -3 & -5 \end{bmatrix}$$

$$\sim_{2R_1+R_2 \to R_2} \begin{bmatrix} -1 & -3 & 2 & 3 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$\sim_{-2R_1 \to R_1} \begin{bmatrix} 2 & 6 & -4 & -6 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$\sim_{3R_2+R_1 \to R_1} \begin{bmatrix} 2 & 0 & -1 & -3 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{-2}R_2 \to R_2 \end{array} \begin{bmatrix} 1 & 0 & -1/2 & -3/2 \\ 0 & 1 & -1/2 & -1/2 \end{bmatrix}$$

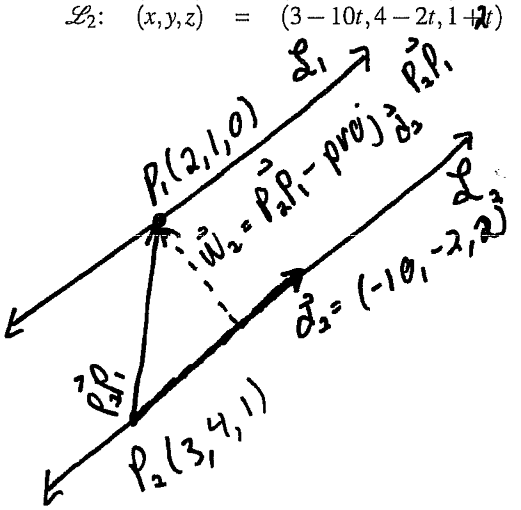
Let $z=t$ sub into

$$\begin{aligned} x - \tfrac{1}{2}z &= -\tfrac{3}{2} \\ y - \tfrac{1}{2}z &= -\tfrac{1}{2} \end{aligned}$$

$$\therefore \quad \begin{aligned} x &= -\tfrac{3}{2} + \tfrac{1}{2}t \\ y &= -\tfrac{1}{2} + \tfrac{1}{2}t \qquad t \in \mathbb{R} \\ z &= t \end{aligned}$$

**Question 6.** *(5 marks)* Using projections find the distance between the following parallel lines

$$\begin{aligned} \mathscr{L}_1: \quad (x,y,z) &= (2+5t, 1+t, -t) \qquad t \in \mathbb{R} \\ \mathscr{L}_2: \quad (x,y,z) &= (3-10t, 4-2t, 1+2t) \qquad t \in \mathbb{R}. \end{aligned}$$




$\therefore d = \|\vec{w}_2\|$

$$\begin{aligned} \vec{P_2P_1} = P_1 - P_2 &= (2,1,0) - (3,4,1) \\ &= (-1,-3,-1) \end{aligned}$$

$$\vec{w}_2 = (-1,-3,-1) - \frac{(-1,-3,-1)\cdot(-10,-2,2)}{(-10,-2,2)\cdot(-10,-2,2)}(-10,-2,2)$$

$$= (-1,-3,-1) - \frac{14}{108}(-10,-2,2)$$

$$= (-1,-3,-1) + \left(\frac{70}{54}, \frac{14}{54}, \frac{-14}{54}\right)$$

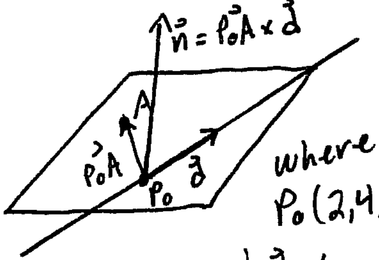
$$= \left(\frac{8}{27}, \frac{-74}{27}, \frac{-34}{27}\right)$$

$$d = \|\vec{w}_2\| = \sqrt{\left(\frac{8}{27}\right)^2 + \left(\frac{-74}{27}\right)^2 + \left(\frac{-34}{27}\right)^2}$$

$$= \sqrt{\frac{6696}{27^2}}$$

$$= \sqrt{\frac{248}{27}}$$

**Question 7.** *(5 marks)* Find the equation of the plane containing the line $(x, y, z) = (2+2t, 4-t, 1-3t) \quad t \in \mathbb{R}$ and the point $A(1, 0, -3)$.




where $P_0(2,4,1)$

and $\vec{d} = (2,-1,-3)$

So $ax + by + cz + d = 0$

$8x - 11y + 9z + d = 0$

sub $(1,0,-3)$

$8(1) - 11(0) + 9(-3) + d = 0$

$d = 19$

$\vec{P_0A} = A - P_0$

$= (1,0,-3) - (2,4,1)$

$= (-1,-4,-4)$

$\vec{n} = \vec{P_0A} \times \vec{d} = \left( \begin{vmatrix} -4 & -1 \\ -4 & -3 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ -4 & -3 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ -4 & -1 \end{vmatrix} \right)$

$\begin{matrix} -1 & 2 \\ -4 & -1 \\ -4 & -3 \end{matrix} = (8, -11, 9)$

$\therefore 8x - 11y + 9z + 19 = 0$

**Question 8.** *(5 marks)* Maximize $Z = 3x + y$ subject to $2x - y \le 60$, $x + y \le 50$.

We have

$$\begin{aligned} 2x - y + S_1 \quad\quad &= 60 \\ x + y \quad + S_2 \quad &= 50 \\ -3x - y \quad\quad + Z &= 0 \end{aligned}$$

pivot

$$\left[\begin{array}{cc:cc:c:c} 2 & -1 & 1 & 0 & 0 & 60 \\ 1 & 1 & 0 & 1 & 0 & 50 \\ \hdashline -3 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} r = \frac{60}{2} = 30 \leftarrow \text{P.r.} \\ r = \frac{50}{1} = 50 \end{array}$$

p.c.

$$\frac{1}{2}R_1 \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 30 \\ 1 & 1 & 0 & 1 & 0 & 50 \\ -3 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{r} \\ -R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 30 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 1 & 0 & 20 \\ 0 & -\frac{5}{2} & \frac{3}{2} & 0 & 1 & 90 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{P.r.} \\ \\ \end{array}$$

p.c.

$$\frac{2}{3}R_2 \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 30 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 & \frac{40}{3} \\ 0 & -\frac{5}{2} & \frac{3}{2} & 0 & 1 & 90 \end{bmatrix}$$

$$\begin{array}{r} \frac{1}{2}R_2 + R_1 \to R_1 \\ \\ \frac{5}{2}R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{110}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 & \frac{40}{3} \\ 0 & 0 & \frac{2}{3} & \frac{5}{3} & 1 & \frac{370}{3} \end{bmatrix}$$

$\therefore x = 110/3 \quad S_1 = 0$

$y = 40/3 \quad S_2 = 0$

$Z = 370/3$

**Question 9.** *(5 marks)* Minimize $Z = x + y$ subject to $x + y \geq 2$, $3x + y \geq 4$. Let $W = -Z = -x-y$

We have

$$\begin{aligned} x + y - S_1 \quad\quad\quad &= 2 \\ 3x + y \quad - S_2 \quad &= 4 \\ x + y \quad\quad\quad + W &= 0 \end{aligned}$$

$$\begin{array}{c} * \\ * \\ \\ \end{array}\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 2 \\ 3 & 1 & 0 & -1 & 0 & 4 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} r = \frac{2}{1} = 2 \\ r = \frac{4}{3} \leftarrow \text{p.r.} \\ \\ \end{array}$$

↑p.c.

$$\tfrac{1}{3}R_2 \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 2 \\ 1 & 1/3 & 0 & -1/3 & 0 & 4/3 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{r} -R_2 + R_1 \to R_1^* \\ \\ -R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 0 & 2/3 & -1 & 1/3 & 0 & 2/3 \\ 1 & 1/3 & 0 & -1/3 & 0 & 4/3 \\ 0 & 2/3 & 0 & 1/3 & 1 & -4/3 \end{bmatrix} \begin{array}{l} r = \frac{2/3}{2/3} = 1 \\ r = 4/3 / 1/3 = 4 \\ \\ \end{array}$$

↑p.c.

$$\tfrac{3}{2}R_1 \begin{bmatrix} 0 & 1 & -3/2 & 1/2 & 0 & 1 \\ 1 & 1/3 & 0 & -1/3 & 0 & 4/3 \\ 0 & 2/3 & 0 & 1/3 & 1 & -4/3 \end{bmatrix}$$

$$\begin{array}{r} \\ -\frac{1}{3}R_1 + R_2 \to R_2 \\ -\frac{2}{3}R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 0 & 1 & -3/2 & 1/2 & 0 & 1 \\ 1 & 0 & 1/2 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$\therefore$ $x = 1$ $\quad S_1 = 0$

$y = 1$ $\quad S_2 = 0$

$Z = -W = -(-2) = 2$

**Bonus Question.** *(5 marks)* Prove: If $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin\theta$. *Hint: Use Lagrange's identity:* $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$.

$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

recall $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

$= \|\vec{u}\|^2\|\vec{v}\|^2 - (\|\vec{u}\|\|\vec{v}\| \cos\theta)^2$

$= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2 \cos^2\theta$

$= \|\vec{u}\|^2\|\vec{v}\|^2 (1 - \cos^2\theta)$

$= \|\vec{u}\|^2\|\vec{v}\|^2 \sin^2\theta$

$\therefore \|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| |\sin\theta|$

$= \|\vec{u}\| \|\vec{v}\| \sin\theta$ since $0 \leq \theta \leq 180°$