Dawson College: **Linear Algebra**: 201-105-dw-S04: Fall 2012

Name: Y. Lamontagne
Student ID:

Test 1

This test is graded out of 49 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(10 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcr} 3x_1 & - & x_2 & + & x_3 & - & 3x_4 & = & 3 \\ 4x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & 5 \\ 7x_1 & + & 2x_2 & & & - & 2x_4 & = & 8 \\ & & & & x_3 & - & x_4 & = & 1 \end{array}$$

$$\begin{bmatrix} 3 & -1 & 1 & -3 & 3 \\ 4 & 3 & -1 & 1 & 5 \\ 7 & 2 & 0 & -2 & 8 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ 3R_2 \\ 3R_3 \\ \\ \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 3 \\ 12 & 9 & -3 & 3 & 15 \\ 21 & 6 & 0 & -6 & 24 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ -4R_1+R_2 \to R_2 \\ -7R_1+R_3 \to R_3 \\ \\ \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 3 \\ 0 & 13 & -7 & 15 & 3 \\ 0 & 13 & -7 & 15 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ -R_2+R_3 \to R_3 \\ \\ \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 3 \\ 0 & 13 & -7 & 15 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ R_3 \leftrightarrow R_4 \\ \\ \end{array} \begin{bmatrix} 3 & -1 & 1 & -3 & 3 \\ 0 & 13 & -7 & 15 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} -R_3+R_1 \to R_1 \\ 7R_3+R_2 \to R_2 \\ \\ \\ \end{array} \begin{bmatrix} 3 & -1 & 0 & -2 & 2 \\ 0 & 13 & 0 & 8 & 10 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} 13R_1 \\ \\ \\ \\ \end{array} \begin{bmatrix} 39 & -13 & 0 & -26 & 26 \\ 0 & 13 & 0 & 8 & 10 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} R_2+R_1 \to R_1 \\ \\ \\ \\ \end{array} \begin{bmatrix} 39 & 0 & 0 & -18 & 36 \\ 0 & 13 & 0 & 8 & 10 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} 1/39\, R_1 \\ 1/13\, R_2 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & -18/39 & 36/39 \\ 0 & 1 & 0 & 8/13 & 10/13 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_4 = t$

then
$x_1 = 36/39 + \frac{18}{39} t$
$x_2 = 10/13 - 8/13\, t$
$x_3 = 1 + t$
$x_4 = t \qquad \forall t \in \mathbb{R}$

Question 2. Consider the matrices:

$$A=\begin{bmatrix}1&-1&2\\0&2&-3\\2&3&4\end{bmatrix},\ B=\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\ C=\begin{bmatrix}0&1\\-3&0\\3&-1\end{bmatrix}\ D=\begin{bmatrix}0&1\\1&-2\end{bmatrix}$$

a. *(2 marks)* Compute the following, if possible.

$C-D$ not possible since dimensions are not the same

b. *(2 marks)* Compute the following, if possible.

$BC-D$

$$BC-D$$

$$=\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\begin{bmatrix}0&1\\-3&0\\3&-1\end{bmatrix}-\begin{bmatrix}0&1\\1&-2\end{bmatrix}$$

$$=\begin{bmatrix}-3&1\\12&0\end{bmatrix}-\begin{bmatrix}0&1\\1&-2\end{bmatrix}$$

$$=\begin{bmatrix}-3&0\\11&2\end{bmatrix}$$

c. *(2 marks)* Compute the following, if possible.

$tr(C)$ not possible since trace is only defined for square matrices

d. *(2 marks)* Compute the following, if possible.

$$tr(A)=tr\left(\begin{bmatrix}1&-1&2\\0&2&-3\\2&3&4\end{bmatrix}\right)=1+2+4=7$$

e. *(3 marks)* Compute the following, if possible.

$$C^tB^t=(BC)^t=\begin{bmatrix}-3&1\\12&0\end{bmatrix}^t=\begin{bmatrix}-3&12\\1&0\end{bmatrix}$$

f. *(2 marks)* Compute the following, if possible.

AB not possible since A is 3×3 and B is 2×3 ($3 \neq 2$)

g. *(5 marks)* Find E, if possible.

$(I-DE)^{-1}=(BC)^t$

$$\left[(I-DE)^{-1}\right]^{-1}=\begin{bmatrix}-3&12\\1&0\end{bmatrix}^{-1}$$

$$I-DE=\frac{-1}{12}\begin{bmatrix}0&-12\\-1&-3\end{bmatrix}$$

$$DE=-\begin{bmatrix}0&1\\1/12&1/4\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$DE=\begin{bmatrix}1&-1\\-1/12&3/4\end{bmatrix}$$

$$D^{-1}DE=D^{-1}\begin{bmatrix}1&-1\\-1/12&3/4\end{bmatrix}$$

$$E=-1\begin{bmatrix}-2&-1\\-1&0\end{bmatrix}\begin{bmatrix}1&-1\\-1/12&3/4\end{bmatrix}$$

$$E=\begin{bmatrix}2&1\\1&0\end{bmatrix}\begin{bmatrix}1&-1\\-1/12&3/4\end{bmatrix}$$

$$=\begin{bmatrix}23/12&-5/4\\1&-1\end{bmatrix}$$

Question 3. *(5 marks)* Given the following augmented matrix in row-echelon form, solve the system using back substitution.

$$\left[\begin{array}{ccccc|c} 1 & 3 & -3 & 1 & 4 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

free variables: x_2, x_5

Let $x_2 = s$
$x_5 = t$

sub into

$$\begin{cases} x_1 + 3x_2 - 3x_3 + x_4 + 4x_5 = 3 \\ x_3 - 2x_4 + 3x_5 = -2 \\ x_4 + 2x_5 = 4 \end{cases}$$

$$\begin{cases} x_1 + 3s - 3x_3 + x_4 + 4t = 3 & (1) \\ x_3 - 2x_4 + 3t = -2 & (2) \\ x_4 + 2t = 4 & (3) \end{cases}$$

So (5) $x_4 = 4 - 2t$ sub into (2)

$x_3 - 2(4-2t) + 3t = -2$

$x_3 = 6 - 7t$ (4)

sub (4) and (5) into (1)

$x_1 + 3s - 3(6-7t) + 4 - 2t + 4t = 3$

$x_1 = 17 - 3s - 23t$

$\therefore (x_1, x_2, x_3, x_4, x_5)$

$= (17 - 3s - 23t, s, 6-7t, 4-2t, t)$

$\forall s,t \in \mathbb{R}$

Question 4. *(5 marks)* Express

$$A = \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} B = \begin{bmatrix} 3 & 4 \\ 8 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}$$

Find the elementary matrices E_1 and E_2 (if possible) such that

a. *(2 marks)* $E_1 A = B$
b. *(2 marks)* $E_2 B = C$
c. *(2 marks)* Are the matrices A and C row equivalent, justify?

a) $A = \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \sim_{R_1+R_2 \to R_2} \begin{bmatrix} 3 & 4 \\ 8 & 2 \end{bmatrix} = B$

So $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim_{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = E_1$

$\therefore E_1 A = B$

b)

$B = \begin{bmatrix} 3 & 4 \\ 8 & 2 \end{bmatrix} \sim_{\frac{1}{2}R_2} \begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix} = C$

So

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim_{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = E_2$

$\therefore E_2 B = C$

c) A and C are row equivalent since

$E_2 E_1 A = C$

equivalent to elem. row op.

Question 5. *(5 marks)* Find the inverse of

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$

$$[A \mid I]$$

$$= \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & -6 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \; 2R_2 \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 6 & 8 & 0 & 0 & 2 & 0 \\ 0 & -6 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \; -3R_1 + R_2 \to R_2 \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 2 & 0 \\ 0 & -6 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \; \begin{array}{r} 3R_2 + R_1 \to R_1 \\ -R_2 \\ -6R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -8 & 6 & 0 \\ 0 & 1 & 0 & 3 & -2 & 0 \\ 0 & 0 & 1 & 18 & -12 & 1 \end{array}\right]$$

$$\sim \; \tfrac{1}{2}R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & 0 \\ 0 & 1 & 0 & 3 & -2 & 0 \\ 0 & 0 & 1 & 18 & -12 & 1 \end{array}\right]$$

$$\therefore \; A^{-1} = \begin{bmatrix} -4 & 3 & 0 \\ 3 & -2 & 0 \\ 18 & -12 & 1 \end{bmatrix}$$

Question 6. A square matrix A is said to be idempotent if $A^2 = A$.

a. *(2 marks)* Show that if A is idempotent, then so is $I - A$.

b. *(3 marks)* Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

a) $(I-A)^2 = (I-A)(I-A) = I \cdot I - I \cdot A - A \cdot I + A \cdot A$
$= I - 2A + A^2$
$= I - 2A + A$ since $A^2 = A$
$= I - A$

b) $(2A-I)(2A-I) = 4A \cdot A - 2A \cdot I - 2I \cdot A + I$
$= 4A^2 - 4A + I$
$= 4A - 4A + I$
$= I$

$\therefore A^{-1} = 2A - I$

Question 7. Consider the following system:

$$\begin{aligned} x + y + 7z &= -7 \\ 3y + 17z &= -16 \\ a(a^2+1)z &= 3a+3 \end{aligned} \equiv \left[\begin{array}{cccc} 1 & 1 & 7 & -7 \\ 0 & 3 & 17 & -16 \\ 0 & 0 & a(a^2+1) & 3a+3 \end{array}\right]$$

where $a \in \mathbb{R}$, determine the values of a so that the system has

a. a unique solution,

b. infitely many solutions,

c. no solutions.

a) has a unique solution if it has 3 leading 1. $\therefore a(a^2+1) \neq 0$

$a \neq 0$

b) has a free variable if it has a row of zero $\therefore a(a^2+1) = 0$

and $3a+3 = 0$

$\therefore$ impossible

c) has a contradiction if

$a(a^2+1) = 0$ & $3a+3 \neq 0$

$\therefore a = 0$

there are no value of a for which the system has infinitely many solutions.

Bonus Question. *(5 marks)* Show that the following nonlinear system has 18 solutions if $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma \le 2\pi$.

$$\begin{array}{rcrcrcl} \sin\alpha & + & 2\cos\beta & + & 3\tan\gamma & = & 0 \\ 2\sin\alpha & + & 5\cos\beta & + & 3\tan\gamma & = & 0 \\ -\sin\alpha & - & 5\cos\beta & + & 5\tan\gamma & = & 0 \end{array} \quad \equiv \quad \begin{array}{rrrl} x & +2y & +3z & = 0 \\ 2x & +5y & +3z & = 0 \\ -x & -5y & +5z & = 0 \end{array}$$

Let $x = \sin\alpha$
$y = \cos\beta$
$z = \tan\gamma$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} -2R_2 + R_1 \to R_1 \\ \\ 3R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} 9R_3 + R_1 \to R_1 \\ -3R_3 + R_2 \to R_2 \\ -R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ $x = 0 = \sin\alpha \therefore \alpha \in \{0, \pi, 2\pi\}$
$y = 0 = \cos\beta \therefore \beta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$
$z = 0 = \tan\gamma \therefore \gamma \in \{0, \pi, 2\pi\}$

$\therefore$ $3 \times 2 \times 3$ solutions