Dawson College: **Linear Algebra**: 201-105-dw-S04: Fall 2012

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Test 2

This test is graded out of 55 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Let

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 3 & 0 \end{bmatrix}$$

a. *(1 mark)* Compute the determinant of A.
b. *(4 marks)* Determine the matrix of cofactors of A.
c. *(1 mark)* Determine $\operatorname{adj}(A)$.
d. *(1 mark)* Determine A^{-1}.
e. *(2 marks)* Solve

$$AX = B$$

for X using the inverse of A.

a) $\det(A) = (-1)(-1)(-1) = -1$

b) $C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} = 0$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$

$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = 2$

$C_{22} = (-1)^{2+2} \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} = 1$

$C_{23} = (-1)^{2+3} \begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3$

$C_{32} = (-1)^{3+2} \begin{vmatrix} -1 & 3 \\ 0 & 0 \end{vmatrix} = 0$

$C_{33} = (-1)^{3+3} \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$

matrix of cofactor $= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

c) $\operatorname{adj} A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$

$= \dfrac{1}{-1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

e) $AX = B$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B = \begin{bmatrix} -1 & -2 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -12 & -6 \\ 0 & -3 \\ -3 & 0 \end{bmatrix}$

Question 2. *(2 marks)* Find the value(s) of λ in order for A to be symmetric

$$A = \begin{bmatrix} 1 & 7 \\ \lambda^2 - 2 & -3 \end{bmatrix}$$

$$A^T = A$$

$$\begin{bmatrix} 1 & 7 \\ \lambda^2-2 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 \\ \lambda^2-2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \lambda^2-2 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ \lambda^2-2 & -3 \end{bmatrix}$$

So $\lambda^2 - 2 = 7$

$\lambda^2 = 9$

$\lambda = \pm 3$

Question 3. *(2 marks)* Find all 5×5 diagonal matrices A that satisfy

$$A^2 - 7A + 10I = 0$$

$$0 = (A - 5I)(A - 2I)$$

$A - 5I = 0$

$A = 5I$

$$= \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$A - 2I = 0$

$A = 2I$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Question 4. *(5 marks)* Find all values of λ for which $\det(A) = 0$.

$$\begin{bmatrix} \lambda-4 & \lambda^{101} & 5 & 0 \\ -1 & 0 & \lambda & 0 \\ 0 & \lambda+2 & 0 & 0 \\ 1 & 2 & 3 & \lambda \end{bmatrix} = A$$

$$|A| = \underbrace{a_{14}}_{0} C_{14} + \underbrace{a_{24}}_{0} C_{24} + \underbrace{a_{34}}_{0} C_{34} + a_{44} C_{44}$$

$$= \lambda (-1)^{4+4} \begin{vmatrix} \lambda-4 & \lambda^{101} & 5 \\ -1 & 0 & \lambda \\ 0 & \lambda+2 & 0 \end{vmatrix}$$

$$= \lambda \left[\underbrace{a_{31}}_{0} C_{31} + a_{32} C_{32} + \underbrace{a_{33}}_{0} C_{33} \right]$$

$$= \lambda (\lambda+2)(-1)^{3+2} \begin{vmatrix} \lambda-4 & 5 \\ -1 & \lambda \end{vmatrix}$$

$$= -\lambda(\lambda+2)\left[(\lambda-4)\lambda + 5\right] = -\lambda(\lambda+2)\left[\lambda^2 - 4\lambda + 5\right]$$

So

$0 = |A|$

$0 = -\lambda(\lambda+2)(\lambda^2 - 4\lambda + 5)$

$\lambda = 0 \quad \lambda = -2$

Question 5. *(6 marks)* Solve *only* for x_2 using Cramer's rule.

$$\begin{array}{rcrcrcrcrcr} 3x_1 & - & x_2 & + & x_3 & - & 3x_4 & + & 4x_5 & = & 2 \\ & & 3x_2 & & & & & - & 10x_5 & = & 3 \\ & & & - & x_3 & - & 2x_4 & + & 3x_5 & = & -3 \\ & & & & & - & x_4 & - & 3x_5 & = & 6 \\ & & & & & & & & 4x_5 & = & 9 \end{array}$$

So $X_2 = \dfrac{|A_2|}{|A|}$

where $|A| = \begin{vmatrix} 3 & -1 & 1 & -3 & 4 \\ 0 & 3 & 0 & 0 & -10 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = 3(3)(-1)(-1)(4) = 36$

and $|A_2| = \begin{vmatrix} 3 & 2 & 1 & -3 & 4 \\ 0 & 3 & 0 & 0 & -10 \\ 0 & -3 & -1 & -2 & 3 \\ 0 & 6 & 0 & -1 & -3 \\ 0 & 9 & 0 & 0 & 4 \end{vmatrix} \begin{array}{c} = \\ R_2+R_3 \to R_3 \\ -2R_2+R_4 \to R_4 \\ -3R_2+R_5 \to R_5 \end{array} \begin{vmatrix} 3 & 2 & 1 & -3 & 4 \\ 0 & 3 & 0 & 0 & -10 \\ 0 & 0 & -1 & -2 & -7 \\ 0 & 0 & 0 & -1 & 17 \\ 0 & 0 & 0 & 0 & 34 \end{vmatrix}$

$= (3)(3)(-1)(-1)(34) = 306$

$\therefore X_2 = \dfrac{|A_2|}{|A|} = \dfrac{306}{36} = \dfrac{17}{2}$

Question 6. *(5 marks)* Evaluate the determinant

$$\begin{vmatrix} d & f & e \\ 2a & 2c & 2b \\ g-4d & i-4f & h-4e \end{vmatrix}$$

given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$$

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\sim \begin{array}{c} R_1 \leftrightarrow R_2 \\ -4R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} d & e & f \\ a & b & c \\ g-4d & h-4e & i-4f \end{bmatrix}$$

$$\sim \quad 2R_2 \to R_2 \begin{bmatrix} d & e & f \\ 2a & 2b & 2c \\ g-4d & h-4e & i-4f \end{bmatrix}$$

$$\overset{C_2 \leftrightarrow C_3}{\sim} \begin{bmatrix} d & f & e \\ 2a & 2c & 2b \\ g-4d & i-4f & h-4e \end{bmatrix} = B$$

det [new matrix] = (op that change det) det [original matrix]

$$\det B = (-1)(2)(-1)(2) = 4$$

Question 7. *(4 marks)* Prove that a square matric A is invertible if and only if A^TA is invertible.

Suppose A is invertible then $\det A \neq 0$. It follows that $\det(A^TA) = \det A^T \det A = \det A \det A \neq 0$ since $\det A \neq 0$. $\therefore$ A^TA is invertible.

Lets now show the converse. That is, suppose A^TA is invertible then

$$\det(A^TA) \neq 0$$
$$\det(A^T)\det A \neq 0$$
$$\det A \det A \neq 0$$

It follows that $\det A \neq 0$. $\therefore$ A is invertible

Question 8. *(5 marks)* Find the determinant given that A and B are 7×7 matrices for which $\det(A) = 3$ and $\det(B) = 5$. *Justify all your work, as usual*

$$\det((2AB^T)^{-1}(3B)A^3) = \det(2AB^T)^{-1} \det(3B) \det A^3$$

$$= \frac{1}{\det(2AB^T)} 3^7 \det B \, [\det A]^3$$

$$= \frac{1}{2^7 \cancel{\det A} \det B^T} 3^7 \det B \, [\det A]^{\cancel{3}\,2}$$

$$= \frac{3^7 [\det A]^2 \cancel{\det B}}{2^7 \cancel{\det B}}$$

$$= \frac{3^7 (3)^2}{2^7}$$

$$= \frac{3}{2^7} = \frac{19683}{128}$$

Question 10. Given

$$A=\begin{bmatrix}0&1&1&1&9&3&4&1&0&9\\0&0&1&1&4&9&2&7&7&9\\0&0&0&1&1&4&9&2&7&7\\0&0&0&0&1&1&4&9&2&7\\0&0&0&0&0&1&1&4&9&2\\0&0&0&0&0&0&1&1&4&9\\0&0&0&0&0&0&0&1&1&4\\0&0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},\ B=\begin{bmatrix}-2&3&2&6\\1&2&3&-3\\-3&0&1&9\\4&5&5&-12\end{bmatrix},C=\begin{bmatrix}1&2&3\\3&1&0\\3&2&1\end{bmatrix}$$

a. *(2 marks)* Is A invertible, justify.

b. *(2 marks)* Is B invertible, justify.

c. *(2 marks)* Compute the determinant of C.

a) A is not invertible since
$\det A = 0(0)(0)(0)(0)(0)(0)(0)(0)(0) = 0$

b) B is not invertible since
$\det B = 0$ since $C_4 = -3C_1$

c) $$\left|\begin{array}{ccc|cc}1&2&3&1&2\\3&1&0&3&1\\3&2&1&3&2\end{array}\right|$$

$= 1\cdot1\cdot1 + 2\cdot0\cdot3 + 3\cdot3\cdot2 - 3\cdot1\cdot3 - 1\cdot0\cdot2 - 1\cdot3\cdot2$

$= 1 + 18 - 9 - 6$

$= 4$

Question 10. Given

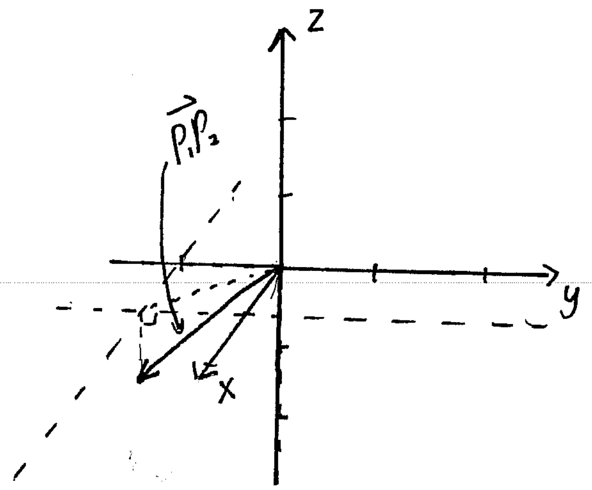
$$P_1(-1,0,1),\ P_2(0,-1,0),\ \mathbf{u}=(1,-2,3),\ \mathbf{v}=(-3,5,1),\ \mathbf{w}=(5,1,-2)$$

a. *(2 marks)* Sketch the vector $\overrightarrow{P_1P_2}$ with the initial point located at the origin.

b. *(2 marks)* Find the initial point of the vector that is equivalent to $\mathbf{u}$ and whose terminal point is P_1.

c. *(2 marks)* Determine $(2\mathbf{u}-7\mathbf{w})-(8\mathbf{v}+\mathbf{u})$.

a) $\vec{P_1P_2} = P_2 - P_1 = (0,-1,0) - (-1,0,1) = (1,-1,-1)$



b) Let I be the initial point

$$\vec{IP_1} = \vec{u}$$

$$P_1 - I = (1,-2,3)$$

$$P_1 - (1,-2,3) = I$$

$$I = (-1,0,+1) - (1,-2,3)$$

$$I = (-2,2,-2)$$

c) $(2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u})$

$= \vec{u} - 7\vec{w} - 8\vec{v}$

$= (1,-2,3) - 7(5,1,-2) - 8(-3,5,1) = (-10,-49,9)$

Bonus Question. *(5 marks)* Given

$$\begin{aligned} b\cos\gamma + c\cos\beta &= a \\ c\cos\alpha + a\cos\gamma &= b \\ a\cos\beta + b\cos\alpha &= c \end{aligned}$$

$$\Leftrightarrow \begin{aligned} c\cos\beta + b\cos\gamma &= a \\ c\cos\alpha \qquad + a\cos\gamma &= b \\ +b\cos\alpha + a\cos\beta \qquad &= c \end{aligned}$$

apply Cramer's rule to show that

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos\alpha = \frac{|A_1|}{|A|} = \frac{ac^2+ab^2-a^3}{2abc} = \frac{a(c^2+b^2-a^2)}{2abc} = \frac{c^2+b^2-a^2}{2bc}$$

where $|A| = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ +b & a & 0 \end{vmatrix} \begin{matrix} 0 & c \\ c & 0 \\ +b & a \end{matrix} = 0\cdot0\cdot0 + abc + abc - (b)(0)(+b) - 0\cdot a\cdot a - 0\cdot cc = 2abc$

and $|A_1| = \begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix} \begin{matrix} a & c \\ b & 0 \\ c & a \end{matrix} = a(0)(0) + cac + bba - b(0)c - a(a)(a) - cb(0) = ac^2 + ab^2 - a^3$